Integrable Spin Systems From Four Dimensions

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A few years ago, Kevin Costello introduced a new approach to integrable spin systems in two dimensions starting from four-dimensional gauge theory. I've lectured about it last summer and you can find the writeup on the arXiv. Today – after a short introduction to be understandable – I will explain more detailed developments in this area that will appear in a forthcoming paper by Costello, Masahito Yamasaki, and me.

Let us start by remembering Chern-Simons gauge theory in three dimensions:

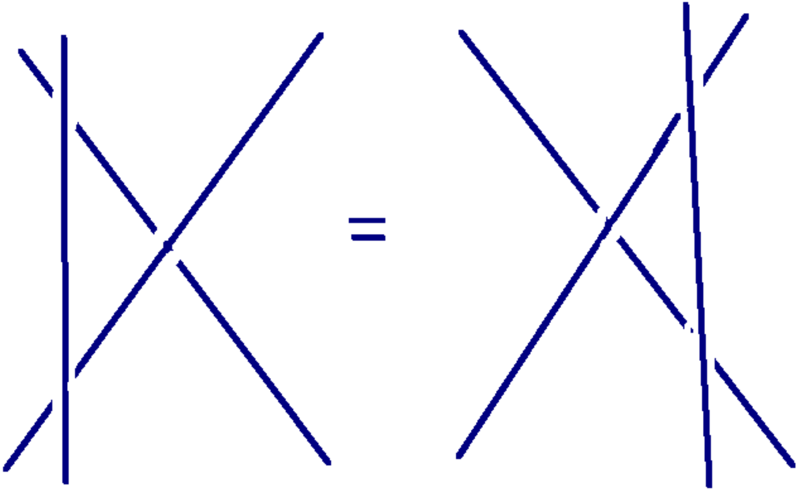
$$I(A) = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \mathrm{d} A + \frac{2}{3} A \wedge A \wedge A \right).$$

There is no metric tensor in sight, so a theory with this action is going to be a topological field theory. In particular, the expectation value of a Wilson line observable

$$W_\rho(K) = \mathrm{Tr}_\rho P \exp \left(\oint_K A \right),$$

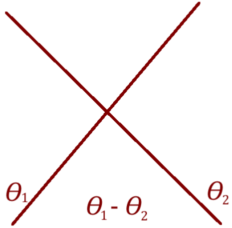
– where ρ is a representation of the gauge group G and K is a knot in spacetime – is a topological invariant.

Knot invariants in particular satisfy the Reidemeister moves, of which the following is the most important one:



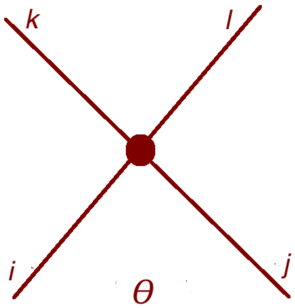
Importantly, this is a fully three-dimensional relation. A given strand passes over or under another.

In the theory of integrable systems, a central role is played by the Yang-Baxter equation. The object that obeys it is called the R-matrix. Before I write the Yang-Baxter equation, I want to introduce the R-matrix that will obey it. It describes the crossing of two lines that you can think of as particle trajectories.



A particle has a "rapidity" or "spectral parameter" but (usually) the R-matrix depends only on the rapidity difference $\theta = \theta_1 - \theta_2$. The physical meaning of this parameter is different for different kinds of integrable systems. (I should point out that when the spectral parameter is treated as a complex number, I will call it z rather than θ.)

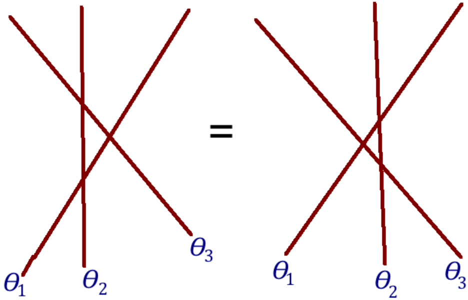
A particle also has an internal label that you can think of as representing a basis vector in a group representation ρ, so the picture looks more like this:



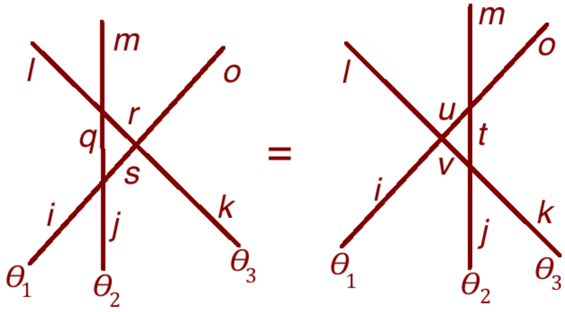
This is a purely two-dimensional picture. There is no notion of one line going “over” or “under” the other.

In Chern-Simons theory, a line (which will be a Wilson line) is labeled by a group representation, but not in any simple way by a basis vector in that representation. As we will see, that is because Chern-Simons theory is scale-invariant and infrared non-free while the Yang-Baxter equation is associated to an infrared-free theory. (In Chern-Simons theory, if one introduces a suitable symmetry breaking mechanism – but it is a long story to do this without spoiling topological invariance – one can get a picture in which a line is labeled by a basis vector and not just a representation.)

The Yang-Baxter equation is the obvious analog of the Reidemeister move that I wrote before except that everything is flattened down to two dimensions and the lines have extra labels, and the picture is completely flat



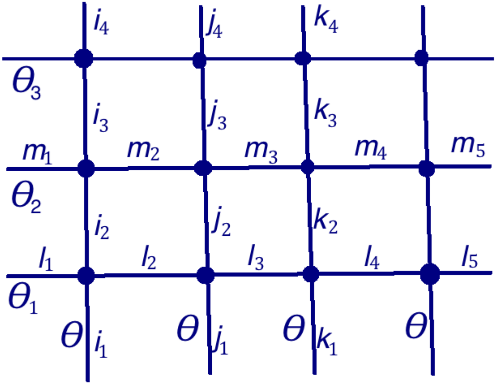
In drawing the picture, to keep things simple, I wrote only the spectral parameters and not the internal labels carried by the particles. A fuller version with all the labels is this:



or schematically

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}.$$

The R-matrix that obeys this equation is the two-particle S-matrix of an integrable many-body system in $1+1$ dimensions, or the four-spin coupling tensor in an integrable spin system:



(The spins live on links, and at every vertex the R-matrix describes the coupling between four spins that meet at that vertex. The partition function is obtained by multiplying all the R-matrix elements and summing over all spins.)

To account for Yang-Baxter theory and the existence of integrable systems, we would like to modify Chern-Simons theory to get a theory (a) with only two-dimensional symmetry and no three-dimensional symmetry, and thus no "over" or "under"; (b) in which a line is labeled by a spectral parameter and an internal label $i, j, k, \cdots$ and not just a group representation.

How can we do this?

A naive idea to get the spectral parameter is to replace the finite-dimensional gauge group G with its loop group $\mathcal{L}G$. We parametrize the loop by an angle θ. The loop group has "evaluation" representations that "live" at a particular value $\theta = \theta_0$ along the loop. (In an evaluation representation, a loop $g(\theta)$ acts via $g(\theta_0)$, for some θ_0. Such simple representations exist because $\mathcal{L}G$ is the loop group and *not* its central extension. The central extension does not have finite-dimensional representations.) We hope that θ will be the spectral parameter carried by a particle in the solution of the Yang-Baxter equation.

Taking the gauge group to be a loop group means that the gauge field $A = \sum_i A_i(x)\mathrm{d}x^i$ now depends also on θ and so is $A = \sum_i A_i(x, \theta)\mathrm{d}x^i$. Note that there is no $\mathrm{d}\theta$ term so this is not a full four-dimensional gauge field. The Chern-Simons action has a generalization to this situation:

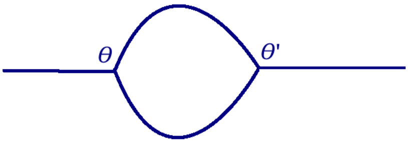
$$I = \frac{k}{4\pi} \int_{M \times S^1} \mathrm{d}\theta \, \mathrm{Tr} \left(A\mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) .$$

This is perfectly gauge-invariant.

What goes wrong is that because there is no $\partial/\partial\theta$ in the action, the "kinetic energy" of A is not elliptic and the perturbative expansion is not well-behaved. The propagator is

$$\langle A_i(\vec{x},\theta)A_j(\vec{x}',\theta') = \frac{\epsilon_{ijk}(x-x')^k}{|\vec{x}-\vec{x}'|^2}\delta(\theta-\theta')$$

with a delta function because the kinetic energy was not elliptic, and because of the delta function, loops will be proportional to $\delta(0)$:



This loop will come with a factor $\delta(\theta-\theta')^2 = \delta(\theta-\theta')\delta(0)$.

What Kevin Costello did was to cure this problem via a very simple deformation. Take our three-manifold to be $\mathbb{R}^3$, and write x, y, t for the three coordinates of $\mathbb{R}^3$, so overall we have x, y, t, and θ. Costello combined t and θ into a complex variable

$$z = \varepsilon t + \mathrm{i}\theta.$$

Here ε is a real parameter. The theory will reduce to the bad case that I just described if $\varepsilon = 0$. As soon as $\varepsilon \neq 0$, its value does not matter and one can set $\varepsilon = 1$. I just included ε to explain in what sense we are making an infinitesimal deformation away from the ill-defined Chern-Simons theory of the loop group.

One replaces $\mathrm{d}\theta$ (or $(k/4\pi)\mathrm{d}\theta$) in the naive theory with $\mathrm{d}z$ (or $\mathrm{d}z/\hbar$) and one now regards A as a partial connection on $\mathbb{R}^3 \times S^1$ that is missing a $\mathrm{d}z$ term (rather than missing $\mathrm{d}\theta$, as before). The action is now

$$I = \frac{1}{\hbar}\int_{\mathbb{R}^3\times S^1} \mathrm{d}z\,\mathrm{Tr}\,\left(A\mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right).$$

(In this form, suitable for perturbation theory only. However, it turns out that perturbation theory converges.)

To keep things simple, we will take z to parametrize the whole complex plane $\mathbb{C}$. This turns out to lead to the rational solutions of the Yang-Baxter equation. (By replacing $\mathbb{C}$ with $\mathbb{C}/\Lambda$ where Λ is a lattice of rank 1 or rank 2, one gets the trigonometric and elliptic solutions of Yang-Baxter.)

Now the first observation about this deformation is that we have lost three-dimensional symmetry – as in knot theory – but we still have two-dimensional symmetry – as in Yang-Baxter theory. Thus we work on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{C}$, where $\mathbb{R}^2$ is parametrized by real coordinates x, y and $\mathbb{C}$ by a complex coordinate z, and we have diffeomorphism symmetry on $\mathbb{R}^2$ (but only translation symmetry along $\mathbb{C}$). So we call $\mathbb{R}^2$ the topological space.

What sort of Wilson lines do we have? We only have a partial gauge field with components A_x, A_y, $A_{\bar{z}}$ and no A_z. So we can do parallel transport in the x and y direction, but we cannot do parallel transport in the z direction. Thus classically a Wilson line has to "live" at a constant value of z. But it can run over an arbitrary curve K in the xy plane, at any fixed value of z:

$$P \exp\left(\int_K (A_x dx + A_y dy)\right).$$

Thus we have only two-dimensional Wilson lines and two-dimensional symmetry, but a Wilson line is labeled by a parameter z that turns out to be the (complexified) spectral parameter. (Some of these statements need to be refined because of a framing anomaly that we discuss later, but it will remain true that a Wilson line depends on a choice of a spectral parameter and a two-dimensional curve.) Later, we will see that there are actually more general Wilson line operators.

We set up perturbation theory in a fairly standard way. We pick a metric on $\mathbb{R}^2 \times \mathbb{C}$, which can be the flat metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + |\mathrm{d}z|^2.$$

Then we fix the gauge $\partial_x A_x + \partial_y A_y + \partial_z A_{\bar{z}} = 0$, leading to a propagator

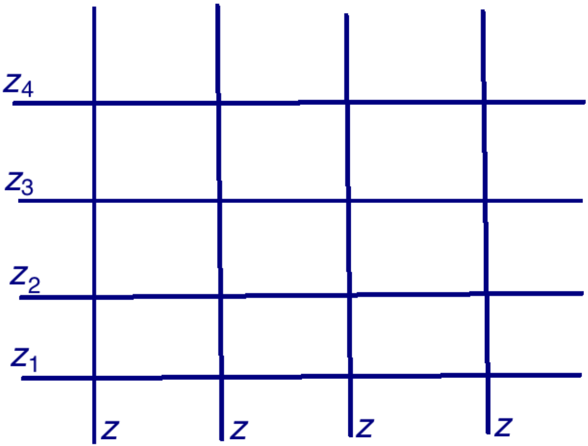
$$\langle A_i(x,y,z) A_j(x',y',z') \rangle = \varepsilon_{ijkz} g^{kl} \frac{\partial}{\partial x^l} \left(\frac{1}{(x-x')^2 + (z-z')^2 + |z-z'|^2} \right)$$

where i, j, k take the values $x, y, \bar{z}$. In contrast to the naive guess we considered first – Chern-Simons theory of the loop group – this is a sensible propagator (no delta functions) and leads to a sensible perturbation expansion.

Perturbation theory on $\mathbb{R}^2 \times \mathbb{C}$ is constructed by expanding around the trivial solution $A = 0$. There are no deformations or automorphisms of this trivial solution and hence the perturbative expansion is straightforward in concept. It gives a simple answer because the theory is infrared-trivial, which is the flip side of the fact that it is unrenormalizable by power-counting. That means that effects at "long distances" in the topological space are negligible.

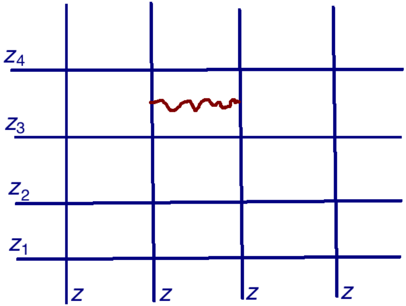
I put the phrase "long distances" in quotes because two-dimensional diffeomorphism invariance means that there is no notion of distance on the topological space $\mathbb{R}^2$ (the first factor of $\mathbb{R}^2 \times \mathbb{C}$). A metric on $\mathbb{R}^2 \times \mathbb{C}$ entered only when we fixed the gauge to pick a propagator. Recall that we used the metric $\mathrm{d}x^2 + \mathrm{d}y^2 + |\mathrm{d}z|^2$. We could equally well scale up the metric along Σ by any factor and use instead $e^B(\mathrm{d}x^2 + \mathrm{d}y^2) + |\mathrm{d}z|^2$ for very large B.

That means that when you look at this picture



you can consider the vertical lines and likewise the horizontal lines to be very far apart (compared to $z - z_i$ or $z_i - z_j$).

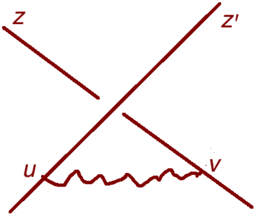
In such a situation, in an infrared-free theory, effects that involve a gauge boson exchange between two nonintersecting lines are negligible:



One should worry about gauge boson exchange from one line to itself

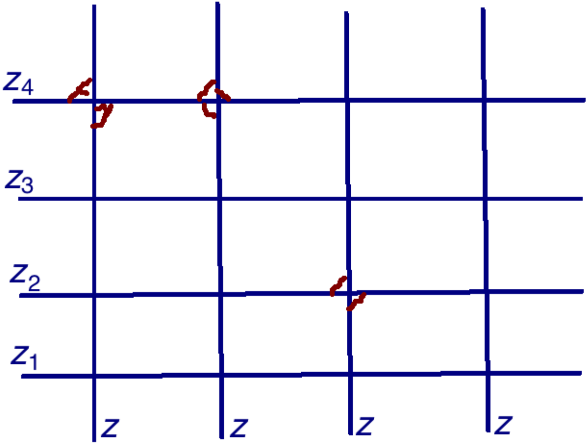
because then the distance $|u - v|$ need not be large. Such effects correspond roughly to "mass renormalization" in standard quantum field theory. In the present problem, the symmetries do not allow any interesting effect analogous to mass renormalization.

When two lines cross we get an integral



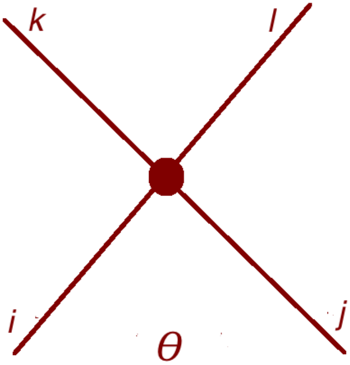
over u and v that converges, and receives significant contributions only from the region $|u|, |v| \lesssim |z - z'|$. I will say what it converges to in a few minutes.

Now when we study a general configuration such as the one related to the integrable lattice models



we can draw very complicated diagrams, but the complications are all localized near one crossing point or another.

Except near a crossing, the gauge field A is effectively 0 (it was 0 classically, and perturbation theory does nothing except near crossings) so a line can be labeled not just by a representation but by a basis vector in that representation. Thus perturbation theory builds up a concrete R-matrix:



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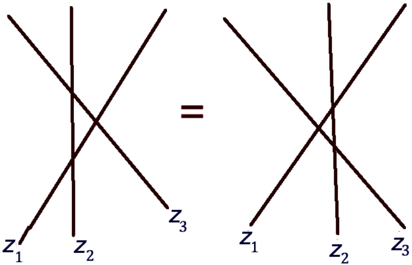
The Yang-Baxter equation



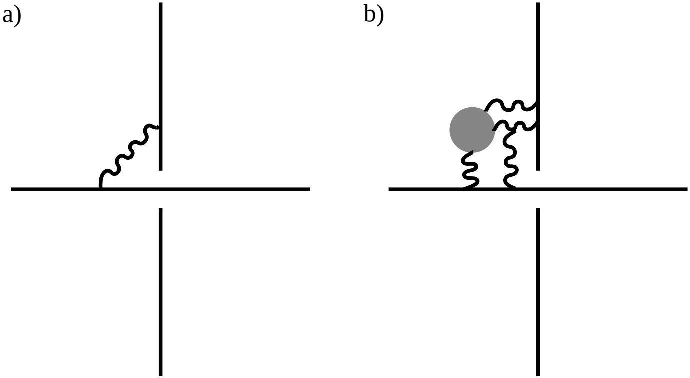
follows from two facts: (1) Two-dimensional diffeomorphism symmetry says that if we move the middle line from left to right, nothing happens except when we try to make it cross through the vertex where the other two lines meet. (2) As long as the spectral parameters (the z_i) are different, two distinct lines are never meeting in four dimensions and thus there can be no discontinuity even when there is such a crossing in the two-dimensional projection.

I should warn you, though, that the flip side of the theory being infrared-free is that it is unrenormalizable by power counting. The theory survives anyway because there are no possible counterterms, but the power counting unrenormalizability makes possible what turns out to be a rather elaborate structure of anomalies that leads to a number of unexpected results. I will try to give an idea of this.

Now I will describe the computations by explicit perturbation theory of three basic effects:

- The lowest order contribution to the R-matrix.
- The lowest order contribution to the fusion of Wilson lines.
- The lowest order contribution to the framing anomaly.

Diagrams that contribute to the R-matrix:

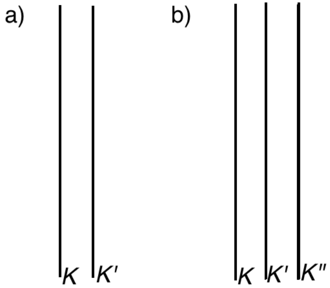


Evaluation of the lowest order diagram is straightforward. Writing $R = 1 + r$ and $z = z_1 - z_2$, and using the formula for the propagator that I wrote before, we get

$$r = \hbar \sum_a \frac{t_a \otimes t_a}{z},$$

which is the standard result for the leading contribution to the r-matrix.

Before discussing the fusion of Wilson lines, I will explain a few generalities about line operators in a topological field theory. In a topological field theory, there is always an OPE of line operators, and it is always associative:



In a diffeomorphism invariant theory, there is no notion of K and K' being "near" or "far," so the product KK' behaves as a single line operator. Associativity is clear. Above 2 dimensions, this OPE would be commutative, but in 2 dimensions in general it is not. We are in that case since the theory under discussion is topological only in 2 dimensions.

However, the statement that there is a closed OPE of line operators will only hold if we include all the line operators of a theory. If we use too small a set of line operators, we will not get a closed OPE: products of line operators may be outside the set that we start with. It turns out in the theory under discussion that the ordinary Wilson line operators I've told you about so far are too small a set.

Assuming that we want line operators that live at $z = 0$, it turns out that there are Wilson line operators associated not just to representations of G but more generally to representations of $\mathfrak{g}[[z]]$. An element of $\mathfrak{g}[[z]]$ is a power series $\sum_{n=0}^{\infty} b_n z^n$, with $b_n \in \mathfrak{g}$. The Lie algebra structure is the obvious one: $[bz^n, b'z^m] = [b, b']z^{n+m}$, as in $1+1$-dimensional current algebra except that we are restricted to $n, m \geq 0$ (there is therefore no central extension). The representations we allow of $\mathfrak{g}[[z]]$ have the property that every vector in the representation is annihilated by bz^n for large enough n (and any b). It is convenient to pick a basis t_a of $\mathfrak{g}$ and to write $t_{a,n}$ for $t_a z^n$. Then the Lie algebra is the familiar

$$[t_{a,n}, t_{b,m}] = f_{abc} t_{c,n+m}.$$

Since the idea of a finite-dimensional representation that is annihilated by bz^n whenever n is large enough may be unfamiliar, I give an explicit example of such a representation of $\mathfrak{g}[[z]]$ that is not just a representation of $\mathfrak{g}$. It will be a representation in which $t_{a,n} = 0$ for $n \geq 2$. We start with any representation R of $\mathfrak{g}$, and then on the direct sum $R \oplus R$, we take

$$t_{a,0} = \begin{pmatrix} t_a & 0 \\ 0 & t_a \end{pmatrix}, \qquad t_{a,1} = \begin{pmatrix} 0 & t_a \\ 0 & 0 \end{pmatrix}, \qquad t_{a,n} = 0,\ \ n \geq 2.$$

What I said before is that an ordinary Wilson line based on a representation of G (or $\mathfrak{g}$) has to live at a point $z \in \mathbb{C}$, which we will take to be $z = 0$. Thus the Line operator

$$P \exp \oint_K A_i \mathrm{d}x^i$$

only depends on the components $A_i(x, y, , z, \bar{z})$ of A. Now informally we can set $\bar{z} = 0$, keeping z fixed, and get a two-dimensional gauge field $A_i(\vec{x}, z, 0)$ (here $\vec{x} = (x, y)$). A precise definition is

$$A_i(\vec{x}, z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \partial_z^k A_i(\vec{x}, z, \bar{z})|_{z=\bar{z}=0}.$$

Here $A_i(\vec{x}, z, 0)$ can be regarded as a gauge field on $\mathbb{R}^2$ with the gauge group being $G[[z]]$. (An important detail is that we do not need to worry about convergence of the series because we will only let this gauge field act on representations of $\mathfrak{g}[[z]]$ in which bz^n acts by 0 for large n, so effectively the series is always truncated after finitely many steps.)

We have the usual gauge invariance $\delta A_i = D_i \varepsilon$, and here we can do the same thing to the gauge parameter ε, setting $\bar{z} = 0$ with fixed z. Thus we define

$$\varepsilon(\vec{x}, z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \partial_z^k \varepsilon(\vec{x}, z, \bar{z})|_{z=\bar{z}=0}.$$

Here $\varepsilon(\vec{x}, z)$ acts as a $\mathfrak{g}[[z]]$-valued gauge parameter for the $\mathfrak{g}[[z]]$ gauge field $A_i(\vec{x}, z)$.

The result of all this is that the theory has line operators

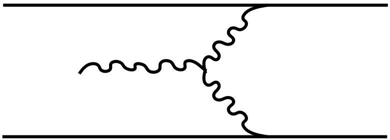
$$P \exp \int_K A_i(\vec{x}, z)$$

for general representations of $\mathfrak{g}[[z]]$ that are not necessarily representations of $\mathfrak{g}$. The reason that this is important is that it turns out that if we consider only the (much) smaller set of representations of $\mathfrak{g}$ only, we do not get a closed OPE.

(I have described line operators supported at $z = 0$. To get line operators supported at $z = z_0$, we similarly consider representations of $\mathfrak{g}[[z - z_0]]$.

Now we are ready to discuss the OPE of two ordinary Wilson lines based on representations of $\mathfrak{g}$.

Here is the lowest order diagram that describes the irreducible coupling of an external gluon to a pair of parallel Wilson lines that I assume come from representations of $\mathfrak{g}$, not $\mathfrak{g}[[z]]$, and both are supported at $z = 0$:



If one evaluates this diagram, in the limit that the spacing ϵ between the two Wilson lines in the topological space $\mathbb{R}^2$ goes to zero, one finds that it contributes a coupling to the composite Wilson line of $\partial_z A$ rather than A. This means that the composite Wilson line is not just associated to a representation of $\mathfrak{g}$. In this order, the composite Wilson line is described by

$$
\begin{aligned}
t_{a,0} &= t_a^1 \otimes 1 \oplus 1 \otimes t_a^2 \\
t_{a,1} &= \hbar f_{abc} t_b^1 \otimes t_c^2 \\
t_{a,n} &= 0, \quad n \geq 2.
\end{aligned} \tag{1}
$$

The formula that I've just written, which comes from the lowest order diagram for the OPE, is one that is frequently encountered in the theory of integrable systems. For example, it describes the action of the first "nonlocal charge" or of the level one generator of the Yangian on two widely separated particles, or on two spins, depending on what kind of integrable system one considers.

This formula, however, implies further deformations, because of the following: the lowest order formula for $t_{a,1}$, namely

$$t_{a,1} = \hbar f_{abc} t_b^1 \otimes t_c^2, \qquad (*)$$

is not consistent with the commutation relations of $\mathfrak{g}[[z]]$. In $\mathfrak{g}[[z]]$, one has

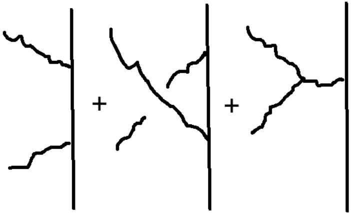
$$[t_{a,1}, t_{b,1}] = f_{abc} t_{c,2}.$$

Therefore, remembering the Jacobi identity, one has

$$f_{uva}[t_{a,1}, t_{b,1}] + \text{cyclic permutations of } u, v, b = 0.$$

The formula $(*)$ is not consistent with this, but the mistake is of order $\hbar^2$, since $t_{a,1}$ is of order $\hbar$.

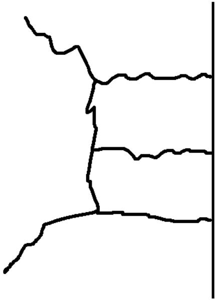
So there will be a further anomaly in order $\hbar^2$ that describes a deformation of the commutation relations of $\mathfrak{g}[[z]]$. To know where to look, one has to remember that in general, to test gauge invariance in Feynman diagrams, one has to look at the coupling of two external gauge bosons to a given charge:



The three diagrams are proportional respectively to $t_a t_b$, $t_b t_a$, and $f_{abc} t_c$. Requiring the sum to be gauge-invariant gives $t_a t_b - t_b t_a - f_{abc} t_c = 0$, in other words the Wilson line transforms in a representation of $\mathfrak{g}$.

So we need to look at diagrams of order $\hbar^2$ that corrects the coupling of a pair of external gauge particles to a given Wilson operator, namely diagrams such as

One.png



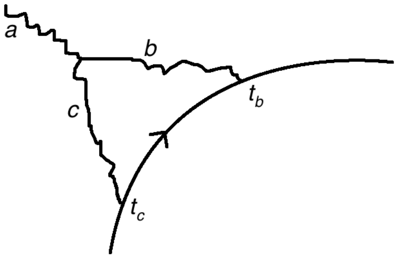
These diagrams do have the expected anomaly. The anomaly deforms $\mathfrak{g}[[z]]$ to the Yangian.

The Yangian is an algebra rather like $\mathfrak{g}[[z]]$ that has generators $t_{a,n}$, where a runs over a basis of $\mathfrak{g}$ and $n = 0, 1, 2, \ldots$, but the commutation relations in the Yangian are more complicated than the $\mathfrak{g}[[z]]$ commutation relations.

In the theory under discussion, at the classical level a Wilson line operator corresponds to a representation of $\mathfrak{g}[[z]]$, but at the quantum level it corresponds to a representation of the Yangian. Not every representation of $\mathfrak{g}$ or more generally of $\mathfrak{g}[[z]]$ can be "deformed" to a representation of the Yangian, and the ones that cannot be so deformed are "lost" in going to the quantum theory.

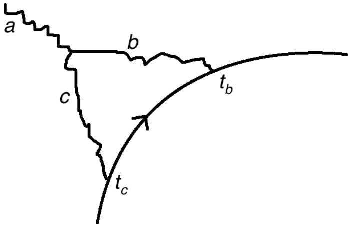
The Yangian is a deformation of the universal enveloping algebra of $\mathfrak{g}[[z]]$, not of $\mathfrak{g}[[z]]$ itself. The reason this happens is that, because the anomaly comes from a diagram with three gluons attached to the given Wilson line, the anomaly is cubic in the group generators of a given representation. Assuming we started with an ordinary representation of $\mathfrak{g}$ (all generators of level 1 and higher were zero at the classical level), one way to cancel the anomaly – if it is not identically 0 for a given representation – is to add level 1 generators acting on the given representation whose commutator is a certain cubic polynomial in the level 0 generators. Because the commutator is cubic, this is a deformation of the universal envelopping algebra, not of $\mathfrak{g}[[z]]$ itself. (One can as well add level 2 generators, also participating in the cancellation of this anomaly.)

The last effect that I will mention today is the framing anomaly, which is responsible for many formulas in the theory of integrable systems that always looked strange to me. Here is the lowest order diagram that gives a quantum correction to the coupling of a single gauge boson to a single Wilson line:



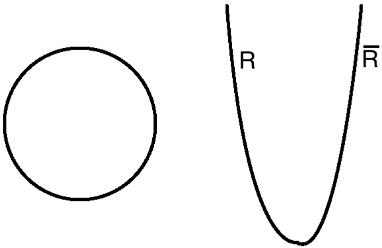
If one evaluates this diagram, one finds the analog in this theory of the usual “framing anomaly” of ordinary Chern-Simons theory.

Consider a Wilson line whose tangent vector is at an angle φ in the xy plane. We are interested in a curved Wilson line, for which φ is non-constant.

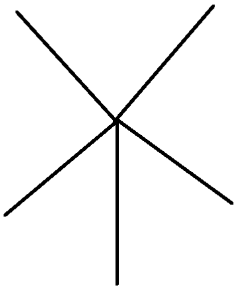


As I've explained before, at the classical level, a Wilson line has $z = \mathrm{constant}$. The diagram I've drawn has an anomaly, such that quantum mechanically what is constant is not z but $z + \hbar \mathsf{h}^\vee \varphi$, where $\mathsf{h}^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

Some applications of the framing anomaly are illustrated here:



For afficionados, another application of the framing anomaly is to the “quantum determinant relation” for SL_N, and analogous relations for other Lie algebras. To understand this relation, one has to first consider “vertices” at which several Wilson lines meet. Here is such a vertex for 5 Wilson lines in the fundamental representation of SL_5:



The framing anomaly comes in if we try to interpret this vertex as determining an invariant vector in the tensor product of five copies of the fundamental representation: