Exact eigenfunctions and the open topological string

Szabolcs Zakany

Department of Theoretical Physics
University of Geneva

Ascona, July 2017

Based on papers : *Marcos Mariño and S.Z. [1606.05297], [1706.07402]*

Introduction

Conjectural correspondence [Grassi-Hatsuda-Mariño],[Codesido-Grassi-Mariño],[Mariño-SZ]:

Spectral theory of operators *spectral determinant, eigenvalues, eigenfunctions*	$\longleftrightarrow$	**Topological strings** *enumerative invariants of target space (toric CY3)*

The target space of the top. string is a toric Calabi-Yau threefold.
The operator is obtained by quantizing the mirror curve.

Provides a concrete and precise non-perturbative completion of topological strings on toric CYs.
Conversely, knowledge of the topological string allows to solve the spectral problem.

In this talk: Focus on extension to open string sector $\rightarrow$ **eigenfunctions**

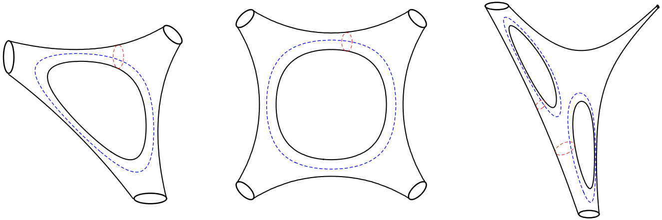
Topological string on toric Calabi-Yau threefold

Consider toric CY three-folds $\mathcal{X}$

A-model: computes Gromov-Witten invariants of $\mathcal{X}$.

B-model: mirror geometry given by a mirror curve in $\mathbb{C}^* \times \mathbb{C}^*$

$0 = W_{\mathcal{X}}(\mathrm{e}^x, \mathrm{e}^y) =$ polynomial of exponentiated coordinates,



local $\mathbb{P}^2$ local $\mathbb{P}^1 \times \mathbb{P}^1$ resolution of $\mathbb{C}^3/\mathbb{Z}_5$

"Canonical" form: $\mathcal{O}(x, y) + \kappa = 0$ κ: modulus (genus 1)

Quantization of the spectral curve

Promote x, y to Heisenberg operators with Weyl ordering

$$\begin{array}{ccc} \mathcal{O}(x,y) & & \mathsf{O} = \mathcal{O}(\mathsf{x},\mathsf{y}) \\ & \longrightarrow & \\ x, y \in \mathbb{C} & & [\mathsf{x},\mathsf{y}] = \mathrm{i}\hbar \end{array}$$

- O is a difference operator. The inverse

$$\rho = \mathsf{O}^{-1}$$

  is a trace class operator on $L^2(\mathbb{R})$

  [Kashaev-Marino], [Laptev-Schimmer-Takhtajan]

  $\rightarrow$ discrete spectrum

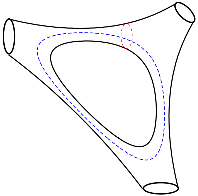
- Important quantity: Fredholm determinant

$$\Xi(\kappa, \hbar) = \det(1 + \kappa\rho)$$

  Its zeros give the spectrum of O: $\qquad \kappa_n, \qquad n = 0, 1, 2, \ldots$

Example: local $\mathbb{P}^2$

Finding spectrum and eigenfunctions of ρ : sharp problem in spectral theory.



Example: toric CY = total space of the canonical bundle over $\mathbb{P}^2$ = "local $\mathbb{P}^2$"

Mirror curve: $\quad e^x + e^y + e^{-x-y} + \kappa = 0$

Operator: $\quad \mathsf{O} = e^{\mathsf{x}} + e^{\mathsf{y}} + e^{-\mathsf{x}-\mathsf{y}}$

Eigenvalues κ_n and eigenfunctions $\psi_n(x)$ can be obtained e.g. numerically.
(Example at $\hbar = 2\pi$):

n	$-\kappa_n$
0	12. 970 039 821...
1	50. 310 469 082...
2	135. 882 402 444...
3	309. 741 176 869...

Conjecture

The conjecture expresses the Fredholm determinant in terms of the enumerative invariants of the underlying CY [Grassi-Hatsuda-Mariño] [Codesido-Grassi-Mariño]. Let us introduce μ related to the modulus $\kappa = \mathrm{e}^{\mu}$.

"Grand potential": $J(\mu, \hbar) = J^{\mathrm{WKB}}(\mu, \hbar) + J^{\mathrm{WS}}(\mu, \hbar)$

- J^{WKB} can be obtained using WKB (resummed in $\hbar$), or the refined topological string in the NS limit.
  [Nekrasov-Shatashvili],[Aganagic-Cheng-Dijkgraaf-Krefl-Vafa]
- $J^{\mathrm{WS}} = F^{\mathrm{GV}}\left(t = \frac{2\pi}{\hbar} t(\mu, \hbar),\ g_s = \frac{4\pi^2}{\hbar}\right)$ ($\to t(\mu, \hbar)$: quantum mirror map)
  free-energies of standard topological string,
  encodes the Gopakumar-Vafa invariants Non-perturbative in $\hbar$.

Standard top. strings provide the non-perturbative $\hbar$ corrections to (resummed) WKB !

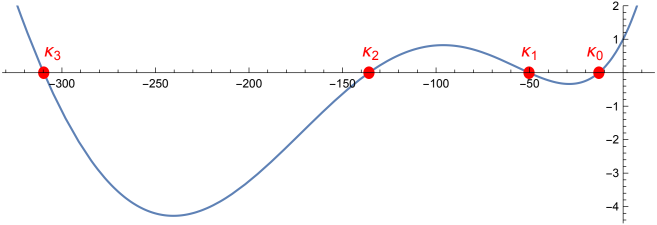
Conjecture

The Fredholm determinant Ξ is given by the Zak transform of J.

$$\text{Conjecture :} \quad \Xi(\kappa, \hbar) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{J(\mu + 2\pi \mathrm{i} n, \hbar)}$$

It is an entire function on the CY moduli space. Gives the spectrum.

Example: $\Xi(\kappa, 2\pi)$ for local $\mathbb{P}^2$



Conjecture for the exact eigenfunctions

The eigenfunction can be studied in the small $\hbar$ limit using the WKB ansatz

$$\psi_{\mathrm{WKB}}(x, \kappa, \hbar) = \exp\left(\frac{\mathrm{i}}{\hbar}\int y(x)\mathrm{d}x + \ldots\right)$$

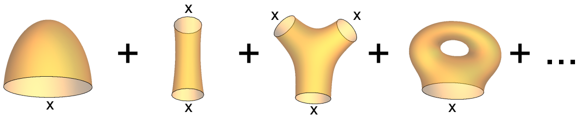
This expression can be resummed in $\hbar$ (like J_{WKB}). We define

$$J^{\mathrm{WKB}}_{\mathrm{open}}(x, \mu, \hbar) = \log \psi_{\mathrm{WKB}}(x)$$

We also need the topological string wavefunction $=$ generating function of D-brane amplitudes, with identification of all the open moduli. It can be written in terms of open-BPS invariants, or using topological recursion

[Bouchard-Klemm-Mariño-Pasquetti]

$$\log \psi_{\mathrm{top}}(x, t, g_s) =$$



Conjecture for the exact eigenfunctions

The non-perturbative part is given by using rescaled variables (like J^{WS})

$$J^{\mathrm{WS}}_{\mathrm{open}}(x, \kappa, \hbar) = \log \psi_{\mathrm{top}} \left(\frac{2\pi}{\hbar} x, \ \frac{2\pi}{\hbar} t(\mu, \hbar), \ g_s = \frac{4\pi^2}{\hbar} \right)$$

Extended "grand potential":

$$J(x, \mu, \hbar) = J^{\mathrm{WKB}}(\mu, \hbar) + J^{\mathrm{WS}}(\mu, \hbar) + J^{\mathrm{WKB}}_{\mathrm{open}}(x, \mu, \hbar) + J^{\mathrm{WS}}_{\mathrm{open}}(x, \mu, \hbar)$$

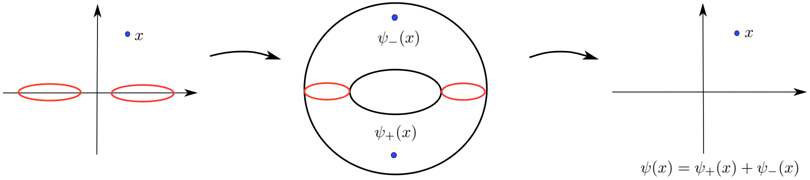
The standard top. string provide the non-perturbative completion to (resummed) WKB.

Extended conjecture: the (unnormalized) eigenfunction is

$$\psi_-(x, \kappa, \hbar) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{J(x, \mu + 2\pi \mathrm{i} n, \hbar)}$$

A subtlety

The function $\psi_-(x,\kappa,\hbar)$ is only one contribution from the wavefunction. It is built from ingredients which are multicovered in x (natural space = the mirror curve). For hyperelliptic curves, there is a second sheet. We obtain $\psi_+(x,\kappa,\hbar)$ by analytically continuing $\psi_-(x,\kappa,\hbar)$ to that second sheet.



$$\psi(x,\kappa,\hbar) = \psi_-(x,\kappa,\hbar) + \psi_+(x,\kappa,\hbar)$$

$\Longrightarrow$ the final eigenfunction is a sum over all the coverings of the closed and open moduli, and is entire in both open and closed moduli

Similar story for the minimal string [Maldacena-Moore-Seiberg-Shih]

Eigenfunctions at $\hbar = 2\pi$

According to these conjectures, $\hbar = 2\pi$ is a special value, where the theory is essentially one loop exact.

Only genus 0 and 1 contributions from closed sector, only disk and annulus amplitudes from the open sector, and only leading and next-to-leading WKB contribute. All are known exactly in x.

$\Longrightarrow \quad \psi_-$ can be written down exactly. It looks like a Baker-Akhiezer function.

$\Longrightarrow \quad \psi_+$ can be worked out by performing the analytic continuation.

Tuning $\kappa = \kappa_n \quad \rightarrow \quad$ "on-shell" eigenfunction.

The on-shell result for local $\mathbb{P}^1 \times \mathbb{P}^1$ has been reproduced from first principles [Kashaev-Sergeev]

Eigenfunctions at $\hbar = 2\pi$

Example: local $\mathbb{P}^2$ (we use $X = \mathrm{e}^x$)

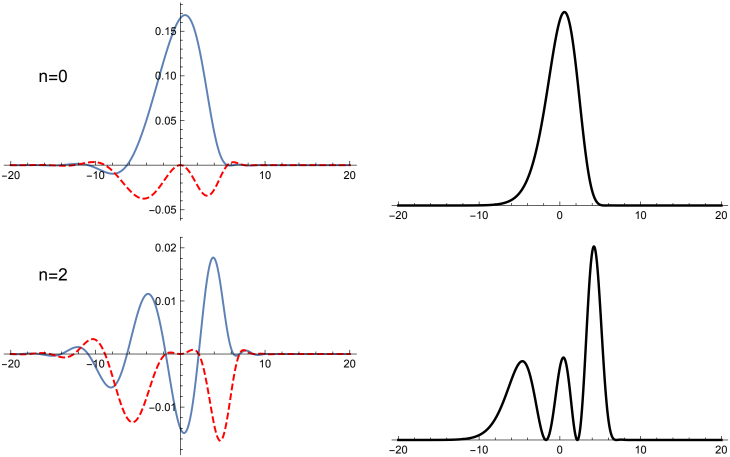
$$\psi(x,\kappa) = \frac{\mathrm{e}^{J(\mu,2\pi)} C \vartheta_1'(0) \mathrm{e}^x}{\sqrt{\sigma(X)}} \left[\mathrm{e}^{-\frac{\mathrm{i}x^2}{2\pi} + \frac{\mathrm{i}}{2\pi}\Sigma(x)} \frac{\vartheta_3(u(X) + \xi - \frac{3}{8})}{\vartheta_1(u(X))} \right.$$
$$\left. + \, \mathrm{e}^{\frac{\pi \mathrm{i}}{4} - \frac{2\pi\mathrm{i}}{3}\xi} \, \mathrm{e}^{\frac{\mathrm{i}x^2}{4\pi} - \frac{\mathrm{i}}{2\pi}\Sigma(x)} \frac{\vartheta_3(u(X) + \xi - \frac{3}{8} + \frac{\tau}{3})}{\vartheta_1(u(X) + \frac{\tau}{3})} \right]$$

- Polynomial $\sigma(X) = X(4 - X(X+\kappa)^2)$,
- $u(X)$ the Abel-Jacobi map based at ∞ and τ the matrix of periods of Riemann surface $\sigma(X) = Y^2$
- τ and ξ are given by $\partial_t F_0$ and its derivative
- $\Sigma(x) = x\tilde{y}(x) - \int^x \mathrm{d}x' \tilde{y}(x') - \frac{2\mathrm{i}\pi}{3} t(2\pi) u(X)$

Eigenfunctions at $\hbar = 2\pi$

The function $\psi(x, \kappa)$ is entire in x for any κ, and normalizable when "on-shell": $\kappa = \kappa_n$, $n = 0, 1, 2, \ldots$

Some eigenfunctions (real and imaginary part, and **(absolute value)^2**):



Application: integrable systems

Cluster integrable systems [Goncharov-Kenyon] : Integrable systems associated to any toric CY3. (Example: relativistic Toda lattice.) The quantized mirror curve gives the analogue of the Baxter equation.

The above results allow to solve the analogue of the Baxter equation.

A modification of the conjecture allows to determined the full exact spectrum of the integrable system [Franco-Hatsuda-Mariño].

Conclusion

- There is a striking but concrete conjectural relationship between the spectral theory of operators on the real line (quantized mirror curves) and enumerative invariants of the underlying toric CYs.
- This provides a non-perturbative definition of topological strings.
- The enumerative invariants allow to solve for the spectrum and other spectral quantities (through the Fredholm determinant)
- Extension of the conjecture relates the eigenfunctions of the operator to open sector of top. strings
  $\Longrightarrow$ allows in some cases to write down exact eigenfunctions
- Interesting link to integrable systems

There are many more things to do ! among others...

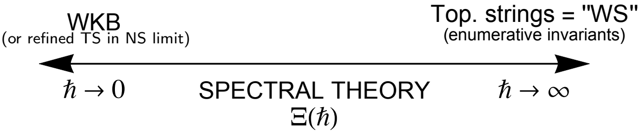
- More general spectral "observables" that encodes the full open string ?
  ($\Longrightarrow$ "non perturbative" full open string)
- Proof ? ...

Thank you !

additional slides...

$$\Xi(\kappa, \hbar) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{J(\mu + 2\pi \mathrm{i} n, \hbar)} \qquad J = J^{\mathrm{WKB}} + J^{\mathrm{WS}}$$

Different limits of the spectral theory:



J^{WS} (=standard topological string free-energies) arise as asymptotic expansion of $\log \Xi$ in the limit $\hbar \to \infty$ ($\mu/\hbar$ finite).

In this limit, WKB part (and effects from the sum) are exponentially suppressed.

Refined NS topological string free energies ($\epsilon_1 = \hbar, \epsilon_2 \to 0$)

$$F^{\mathrm{NS}}(\mathbf{t}, \hbar) = \frac{1}{6\hbar} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{i=1}^{s} b_i^{\mathrm{NS}} t_i \hbar + \sum_{j_L, j_R} \sum_{w, \mathbf{d}} N^{\mathbf{d}}_{j_L, j_R} \frac{\sin \frac{\hbar w}{2}(2j_L+1) \sin \frac{\hbar w}{2}(2j_R+1)}{2w^2 \sin^3 \frac{\hbar w}{2}} \mathrm{e}^{-w\mathbf{d}\cdot\mathbf{t}}$$

$$\mathsf{J}_S^{\mathrm{WKB}}(\mu, \boldsymbol{\xi}, \hbar) = \sum_{i=1}^{s} \frac{t_i(\hbar)}{2\pi} \frac{\partial F^{\mathrm{NS}}(\mathbf{t}(\hbar), \hbar)}{\partial t_i} + \frac{\hbar^2}{2\pi} \frac{\partial}{\partial \hbar} \left(\frac{F^{\mathrm{NS}}(\mathbf{t}(\hbar), \hbar)}{\hbar} \right) + \sum_{i=1}^{s} \frac{2\pi}{\hbar} b_i t_i(\hbar) + A(\boldsymbol{\xi}, \hbar)$$

Standard topological string free-energies (GV)

$$F^{\mathrm{GV}}(\mathbf{t}, g_s) = \sum_{g \geq 0} \sum_{\mathbf{d}} \sum_{w=1}^{\infty} \frac{1}{w} n_g^{\mathbf{d}} \left(2 \sin \frac{w g_s}{2} \right)^{2g-2} \mathrm{e}^{-w\mathbf{d}\cdot\mathbf{t}}$$

$$\mathsf{J}^{\mathrm{WS}}(\mu, \hbar) = F^{\mathrm{GV}} \left(\frac{2\pi}{\hbar} \mathbf{t}(\hbar) + \pi \mathrm{i} \mathbf{B}, g_s = \hbar_{\mathrm{D}} \equiv \frac{4\pi^2}{\hbar} \right)$$

Resummed WKB wavefunction ($X = \mathrm{e}^x$)

$$J^{\mathrm{WKB}}_{\mathrm{open}}(\mu, \hbar, X) = \log \psi_{\mathrm{WKB}}(x)$$

$$= J^{\mathrm{WKB}}_{\mathrm{pert}}(\mu, \hbar, X) + \sum_{\mathbf{d}, \ell, s} \sum_{k=1}^{\infty} D^{s}_{\mathbf{d}, \ell} \frac{q_{\hbar}^{ks}}{k(1 - q_{\hbar}^{k})} (-X)^{-k\ell} \mathrm{e}^{-k\mathbf{d}\cdot\mathbf{t}}$$

Topological string wavefunction (g_s expansion)

$$\psi_{\mathrm{top}}(X, \mathbf{t}, g_s) \sim \exp\left[\sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \frac{(-\mathrm{i} g_s)^{2g-2+h}}{h!} \int^{X} \cdots \int^{X} W_{g,h}(X_1, \cdots, X_h) \mathrm{d}X_1 \cdots \mathrm{d}X_h\right]$$

Resummed topological string wavefunction ($X = \mathrm{e}^x$)

$$\log \psi_{\mathrm{top}}(X, \mathbf{t}, g_s) = \sum_{\mathbf{d}} \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \sum_{\boldsymbol{\ell}} \sum_{w=1}^{\infty} \frac{\mathrm{i}^h}{h!} n_{g,\mathbf{d},\boldsymbol{\ell}} \frac{1}{w} \left(2 \sin \frac{w g_s}{2}\right)^{2g-2}$$
$$\times \prod_{i=1}^{h} \left(2 \sin \frac{w \ell_i g_s}{2}\right) \frac{1}{\ell_1 \cdots \ell_h} X^{-w(\ell_1 + \cdots + \ell_h)} \mathrm{e}^{-w\mathbf{d}\cdot\mathbf{t}}.$$

$$J^{\mathrm{WS}}_{\mathrm{open}}(\mu, \hbar, X) = \log \psi_{\mathrm{top}}\left(X^{2\pi/\hbar}, \frac{2\pi}{\hbar}\mathbf{t}(\hbar) + \pi \mathrm{i} \mathbf{B}, \hbar_{\mathrm{D}}\right).$$