Dipole CFT and Integrability

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M. Guica, F. Levkovich-Maslyuk, K.Z., 1706.07957

String Theory and Quantum Gravity, Ascona, 5.7.17

Non-relativistic holography

Schrödinger background:

$$ds^2 = -\frac{\ell^2 du^2}{z^4} + \frac{-2du\,dv + d\mathbf{x}^2 + dz^2}{z^2}$$

Son'08

Balasubramanian,McGreevy'08

Scale-invariant but time and space scale differently:

$$\begin{array}{ccc} u & \to & \lambda^2 u \\ z & \to & \lambda z \\ \mathbf{x} & \to & \lambda \mathbf{x} \end{array}$$

CFT length scale

$$\ell = \frac{\sqrt{\lambda}}{2\pi} L$$

AdS length scale

Holographic dictionary: $m_\phi \longleftrightarrow \Delta$

$$\phi(v,u,\mathbf{x},z) \sim \mathrm{e}^{\,iMv}\phi(u,\mathbf{x},z)$$

Solving Klein-Gordon equation:

$$\begin{aligned}\Delta &= \frac{d}{2}+\sqrt{\frac{d^2}{4}+m_\phi^2+\ell^2M^2}\\ &= \frac{d}{2}+\sqrt{\frac{d^2}{4}+m_\phi^2+\frac{\lambda}{4\pi^2}L^2M^2}\end{aligned}$$

Volovich,Wen'08
Blau,Hartong,Rollier'09

not protected!

NR CFT & DLCQ

$$P_- \left|M\right\rangle = M \left|M\right\rangle \qquad \Longrightarrow \qquad P_+ = \frac{\mathbf{P}^2}{2M}$$

Unbroken symmetry (Schrödinger algebra):

$$\mathfrak{Sch} = \{T_a \in \mathfrak{so}(d,2) | [T_a, P_-] = 0\}$$

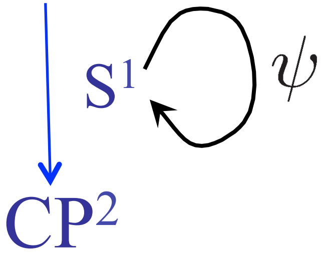
$$\langle \underbrace{P_+, P_i, L_{ij}, L_{i-};}_{\text{Galilean}} \; D - L_{+-}; \; K_-; \; M \rangle$$

NR dilatation

Mass

Schrödinger from light-cone TsT

TsT transformation of $AdS_5 x S^5$:



$$
\begin{aligned}
\mathrm{T-duality}: &\quad \psi \to \tilde{\psi} \\
\mathrm{shift}: &\quad v \to v + \ell \tilde{\psi} \\
\mathrm{T-duality}: &\quad \tilde{\psi} \to \tilde{\tilde{\psi}}
\end{aligned}
$$

$$
\begin{aligned}
ds^2 &= -\frac{\ell^2 du^2}{z^4} + \frac{-2du\,dv + d\mathbf{x}^2 + dz^2}{z^2} + d\Omega^2_{S^5} \\
F_5 &= (1+*)\,\mathrm{Vol}(S^5) \\
B &= \frac{1}{z^2}\,du \wedge \eta
\end{aligned}
$$

Herzog,Rangamani,Ross'08
Maldacena,Martelli,Tachikawa'08
Adams,Balasubramanian,McGreevy'08

vertical 1-form of Hopf fibration

Frolov'05
Frolov,Roiban,Tseytlin'05

TsT ⇔ Twisted boundary conditions:

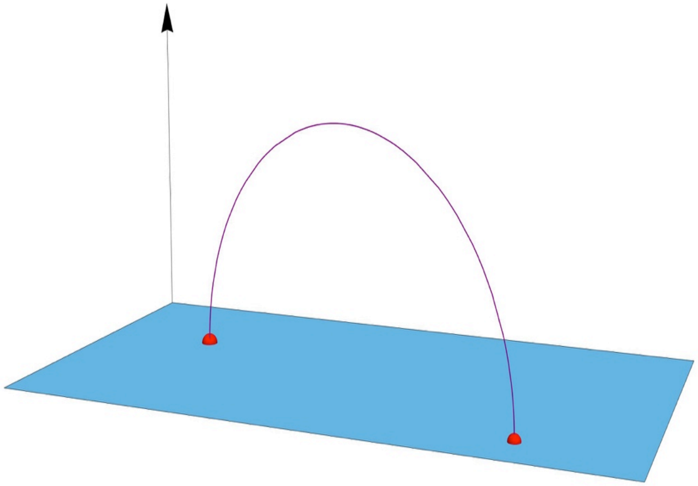
total ang. momemtum

$$\begin{aligned} v(\sigma+2\pi,\tau) &= v(\sigma,\tau)+LJ \\ \psi(\sigma+2\pi,\tau) &= \psi(\sigma,\tau)-LM \end{aligned}$$

total light-cone momemtum

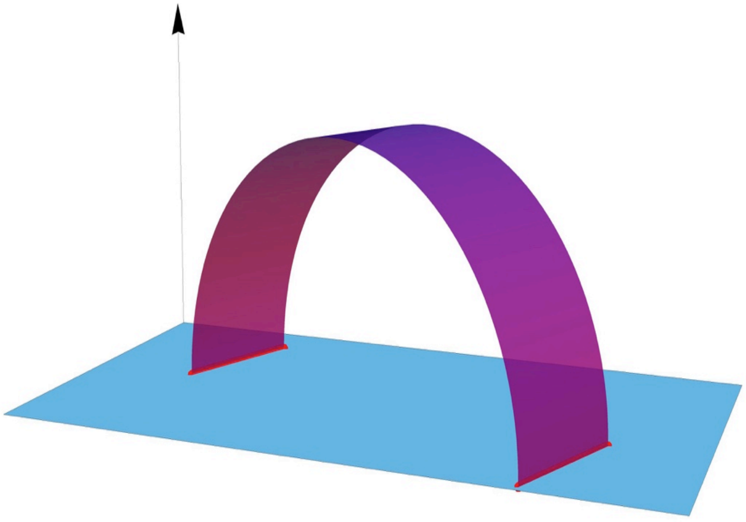
Example: BMN string

Closed-string b.c.'s:



$$\Delta = J$$

Twisted b.c.'s:

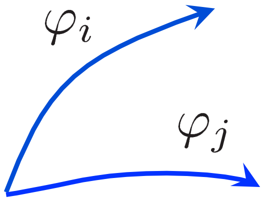


$$\Delta = \sqrt{J^2 + \frac{\lambda}{4\pi^2} L^2 M^2}$$

- dual op. is non-local
- has anomalous dim.

AdS/CFT triality

Lunin,Maldacena'05
Beisert,Roiban'05
Frolov'05

N=4 SYM	AdS$_5$xS^5	Spin chain
Star Product	**TsT**	**Drinfeld-Reshetikhin twist**
$A*B = \mathrm{e}^{\frac{i}{2}\gamma_{ij}Q^i_A Q^j_B} AB$ commuting charges	 commuting isometries	$h_{ab} \to F_{ab} h_{ab} F_{ba}$ $F_{ab} = \mathrm{e}^{\frac{i}{2}\gamma_{ij}Q^i_a Q^j_b}$ Cartan generators

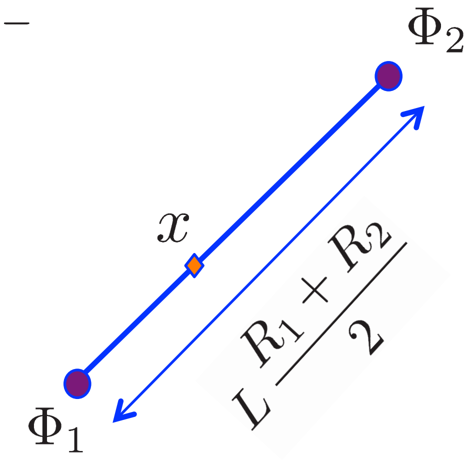
Dipole CFT

Bergman,Ganor'00

Star product:

$$\Phi_1 * \Phi_2(x) = \; \mathrm{e}^{\frac{L}{2}(\partial_{-1}R_2 - R_1\partial_{-2})}\Phi_1(x_1)\Phi_2(x_2)\Big|_{x_{1,2}=x} = \Phi_1(x+L_2)\Phi_2(x-L_1)$$

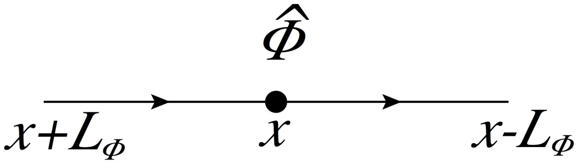
Dipole length: $L_a = \frac{L}{2} R_a n_-$



$$\mathcal{L} = \frac{2}{g^2}\,\mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu Z_j^\dagger * \mathcal{D}^\mu Z_j - \frac{1}{2}[Z_j^\dagger, Z_j]_*^2 + [Z_j^\dagger, Z_k^\dagger]_*[Z_j, Z_k]_*\right)$$

Seiberg-Witten map

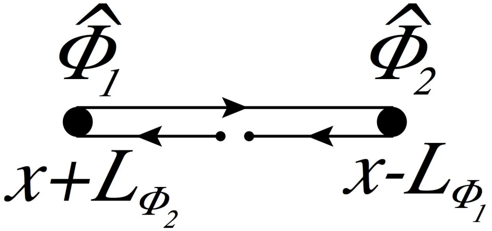
$$\Phi(x) \to U^{-1} * \Phi * U(x) = U^{-1}(x + L_\Phi)\Phi(x)U(x - L_\Phi)$$



$$\Phi(x) = [x + L_\Phi, x]\hat{\Phi}(x)[x, x - L_\Phi], \qquad [x, y] = \mathrm{P}\exp\int_x^y dv\, A_-(v)$$

transforms locally

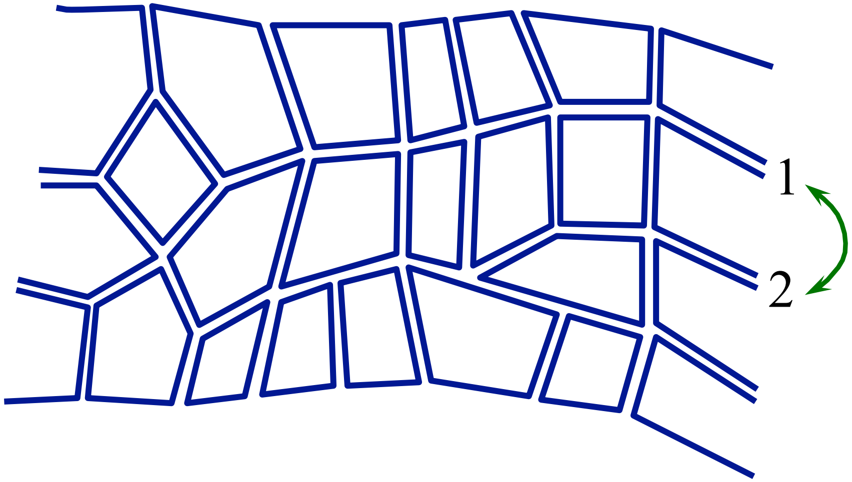
$$\hat{\Phi}_1 \hat{*} \hat{\Phi}_2 = \mathrm{e}^{\frac{iL}{2}(P_{-\,1}R_2 - R_1 P_{-\,2})}\hat{\Phi}_1\hat{\Phi}_2, \qquad P_\mu = -iD_\mu$$



Planar equivalence

$\Phi_1\Phi_2 \rightarrow \Phi_1 * \Phi_2 \quad \Longrightarrow$ Phase factors in interaction vertices

Equivalence theorem: phases cancel in planar diagrams (or can be moved onto external lines)



$$e^{\frac{iL}{2}(p_{-\,1}R_2 - p_{-\,2}R_1)}$$

Operators and spin chains

$$\mathcal{O} = \mathrm{tr}\,\hat{\Phi}_1 \ldots \hat{\Phi}_J$$

- Hamiltonian is twisted by a phase
- Boundary conditions are periodic

$$\mathcal{O} = \mathrm{tr}\,\Phi_1 * \ldots * \Phi_J$$

- Hamiltonian is the same as in N=4 SYM
- Boundary conditions are twisted

Drinfeld-Reshetikhin twist

Drinfeld'90
Reshetikhin'90

$$H = \sum_{l=1}^{J} h_{l,l+1}$$

$$\tilde{h}_{ab} = F_{ab} h_{ab} F_{ba}$$

Beisert,Roiban,05
Ahn,Bajnok,Bombardelli,Nepomechie'10

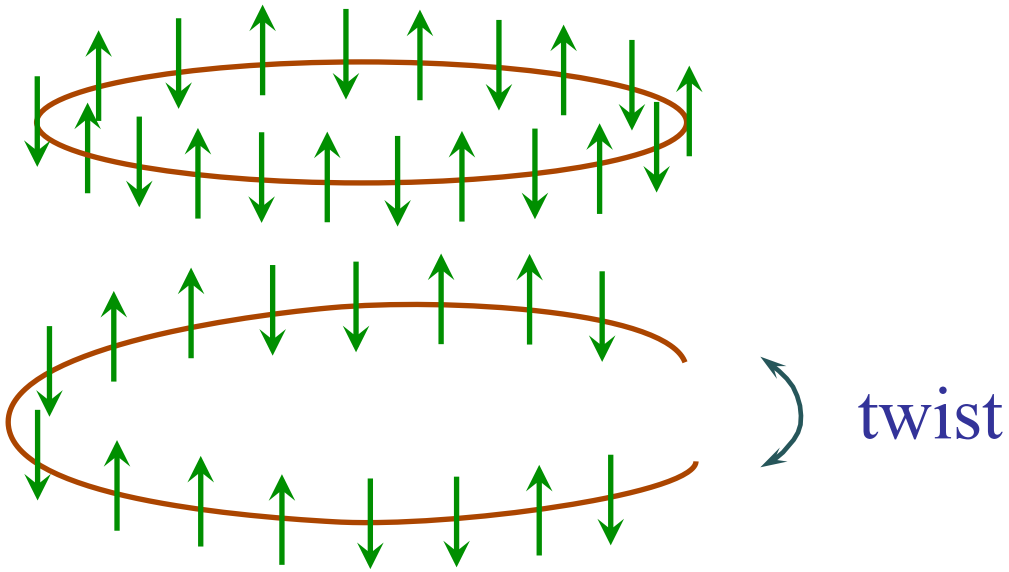
$$F_{ab} = \mathrm{e}^{\frac{iL}{2}(P_{-\,a} R_b - R_a P_{-\,b})}$$

Model remains integrable:

$$\tilde{L}_{al}(u) = F_{al} L_{al}(u) F_{al}$$

satisfies YBE with the same R-matrix

Twist can be transferred from Hamiltonian to boundary conditions:



$$\mathcal{O} = \mathrm{tr}\,\hat{\Phi}_1 \ldots \hat{\Phi}_J$$

$$\tilde{h}_{l,l+1}$$

$$\mathcal{O} = \mathrm{tr}\,\Phi_1 * \ldots * \Phi_J$$

$$h_{l,l+1}$$

closed
strings

twisted
B.C.'s

$$\begin{aligned} \varphi_j(\tau,\sigma+2\pi)-\varphi_j(\tau,\sigma) &= -LM \\ v(\tau,\sigma+2\pi)-v(\tau,\sigma) &= LR \end{aligned}$$

Frolov'05
Frolov,Roiban,Tseytlin'05

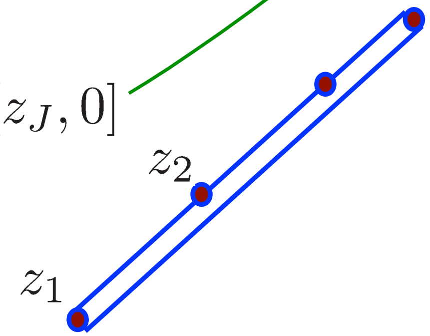
sl(2) sector

$$\mathcal{O} = \mathrm{tr}\, D_-^{S_1} Z \dots D_-^{S_J} Z$$

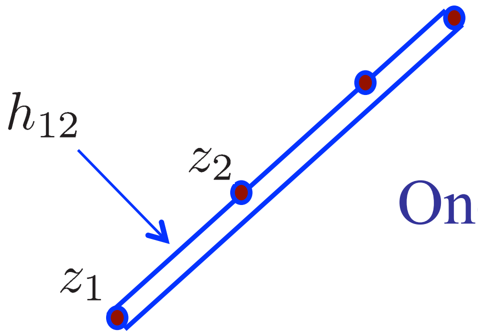
Taylor expansion

- Undeformed theory: finite-dim mixing of local ops
- Dipole CFT:
    - mixing: quantum-mechanical problem
    - eigenstates: non-local ops

$$\mathcal{O}(z_1, \dots, z_J) = \mathrm{tr}[0, z_1] Z(z_1) [z_1, 0] \dots [0, z_J] Z(z_J) [z_J, 0]$$



Spin chain



One-loop dilatation operator: $D = \sum_{\ell=1}^{L} h_{\ell,\ell+1}$

Untwisted:

$$h_{ab}\mathcal{O}(z_a, z_b) = \frac{\lambda}{8\pi^2}\int_0^1 \frac{du}{u}\ \left(2\mathcal{O}(z_a, z_b) - \mathcal{O}(z_a + uz_{ba}, z_b) - \mathcal{O}(z_a, z_b + uz_{ab})\right)$$

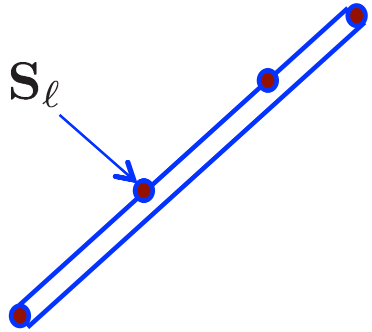
Balitsky,Braun’89

Belitsky,Derkachov,Korchemsky,Manashov’04

Twisted:

$$h_{ab}\mathcal{O}(z_a, z_b) = \frac{\lambda}{8\pi^2}\int_0^1 \frac{du}{u}\ \left(2\mathcal{O}(z_a, z_b) - \mathcal{O}(z_a + uz_{ba} \boxed{+ uL}, z_b) - \mathcal{O}(z_a, z_b + uz_{ab} \boxed{- uL})\right)$$

Algebraic Bethe Ansatz



Spin ½ representation of sl(2) at each site:

$$S^0 = z\partial + \frac{1}{2}, \qquad S^- = \partial, \qquad S^+ = z^2\partial + z$$

Lax operator (2x2 matrix):

$$L_\ell(u) = u + i\boldsymbol{\sigma} \cdot \mathbf{S}_\ell$$

Monodromy matrix:

$$T(u) = L_1(u) \ldots L_J(u)$$

Satisfy YBE

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$|0\rangle = \delta(z_1)\dots\delta(z_J) \qquad \Longleftrightarrow \qquad \mathrm{tr}\, Z^J(0)$$

$$C(u)\,|0\rangle = 0$$

Eigenstates: $B(u_1)\dots B(u_S)\,|0\rangle$

Bethe equations: $$\left(\frac{u_j + \frac{i}{2}}{u_j + \frac{i}{2}}\right)^J = \prod_{k\neq j} \frac{u_j - u_k - i}{u_j - u_k + i}$$

Anomalous dimension: $$\Delta = J + S + \frac{\lambda}{8\pi^2} \sum_j \frac{1}{u_j^2 + \frac{1}{4}}$$

Drinfeld-Reshetikhin twist

$$\tilde{T}(u) = L_1(u) \dots L_J(u) K$$

$$K = \mathrm{e}^{L\sigma^-} = \begin{pmatrix} 1 & 0 \\ LJ & 1 \end{pmatrix}$$

Rep. of P_ in auxiliary space:

$$\tilde{T}(u) = \begin{pmatrix} A(u) + LJB(u) & B(u) \\ C(u) + LJD(u) & D(u) \end{pmatrix}$$

$$(C(u) + LJD(u)) \left|\Omega\right\rangle = 0 \qquad \forall u$$

doesn't exist

No Bethe ansatz!

Baxter equation

$$\left(\frac{u_j + \frac{i}{2}}{u_j + \frac{i}{2}}\right)^J = -\prod_k \frac{u_j - u_k - i}{u_j - u_k + i}$$

Baxter polynomial: $Q(u) = \prod_j (u - u_j)$

$$(u + i/2)^J Q(u + i) + (u - i/2)^J Q(u - i) \equiv P(u)$$

Bethe eqns => $P(u_j) = 0$ => $P(u) = t(u)Q(u)$

Baxter equation:

$$\boxed{(u+i/2)^J Q(u+i) + (u-i/2)^J Q(u-i) = t(u)Q(u)}$$

$$t(u) = 2u^J + \ldots$$

$$\Delta^{1-\mathrm{loop}} = \frac{i\lambda}{8\pi^2}\,\partial_u \log \frac{Q(u+i/2)}{Q(u-i/2)}\bigg|_{u=0}$$

Untwisted theory:

$t(u) = 2u^J + 0u^{J-1} + \ldots$

Q(u) polynomial

Twisted theory:

$t(u) = 2u^J + LMu^{J-1} + \ldots$

Q(u) non-polynomial

Example: twist-2 operators

$$\mathcal{O} = \mathrm{tr}\, Z D^S Z$$

$$(u+i/2)^J Q(u+i) + (u-i/2)^J Q(u-i) = (2u^2 + LMu + t_0) Q(u)$$

Mellin transform:

Faddeev,Korchemsky'93

$$Q(u) = \cosh \pi u \int_{-L}^{L} dz\ (L-z)^{-iu-\frac{1}{2}}\ (L+z)^{iu+\frac{1}{2}}\ \mathrm{e}^{-iMz} \psi\left(\frac{z}{L}\right)$$

Coincides with coordinate-space wave function

$$(1-x^2)\psi'' - 2x\psi' + (L^2M^2 - x^2)\psi = (1/2 + t_0)\psi$$

Solution at M=0:

Untwisted

$$\psi(z) = \left(\frac{d}{dz}\right)^S \delta(z)$$

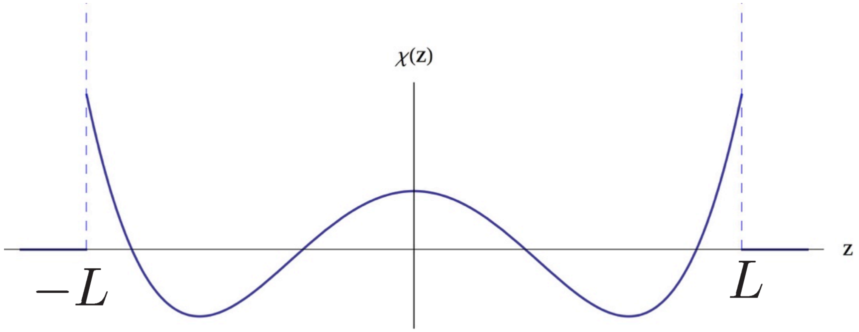
Eigenstates – local ops. $\mathrm{tr}\, Z D_-^S Z$

Twisted

$$\psi(z) = \sqrt{L}\, P_S(z/L) \times \mathrm{Step}(z/L)$$

Legendre polynomials

Eigenstates – non-local



Eigenvalues:

$$\delta\Delta = \frac{\lambda}{2\pi^2}\, h(S)$$

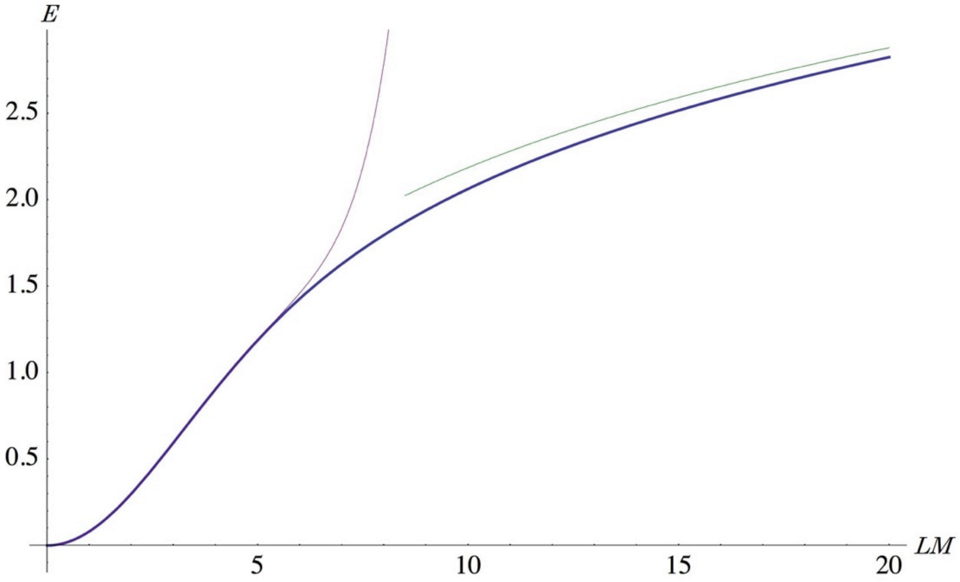
General solution

$$(1-x^2)\psi'' - 2x\psi' + (L^2M^2 - x^2)\psi = (1/2 + t_0)\psi$$

Spheroidal wave equation

$$t_0 + 1/2 = -\lambda_{S,0}\left(\frac{LM}{2}\right)$$

$$\psi(x) = \mathrm{PS}_{S,0}\left(\frac{LM}{2}, x\right)$$



Non-zero momentum

- States with different S mix
- Ground state energy is lifted

To leading order in LM:

$$\delta\Delta_0 = \frac{\lambda L^2 M^2}{24\pi^2} + \mathcal{O}(L^4 M^4)$$

More general result for any J:

$$\delta\Delta_0 = \frac{\lambda L^2 M^2}{8\pi^2(J+2)} + \mathcal{O}(L^4 M^4)$$

Agrees with BMN string energy at J>>1:

$$\Delta = \sqrt{J^2 + \frac{\lambda}{4\pi^2} L^2 M^2}$$

Conclusions

- Dipole CFT is integrable at planar level
- Full system of TQ (or QQ) equations at one loop?
- Quantum Spectral Curve:
  - ✓ same as in N=4 SYM, only asymptotics of Q-functions is different
- Dipole deformations of AdS_3/CFT_2

Azeyanagi,Hofman,Song,Strominger'12

Spinning BHs, Kerr/CFT,…