Gluon Scattering in AdS from CFT

Luis Fernando Alday

University of Oxford

Strings, Fields and Holograms @ Ascona

What will this talk be about?

Conformal Field Theories techniques to study String/M-theory scattering amplitudes on AdS

Based on

- L.F.A. and X. Zhou [arXiv:2006.06653] and [arXiv:2006.12505].
- L. F. A., C. Behan, P. Ferrero and X. Zhou [arXiv:2103.15830].

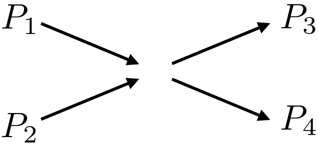
Organisation

1. Scattering of gravitons (closed strings) and main techniques.
2. Scattering of gluons (open strings).

Scattering amplitudes

Scattering Amplitudes

Probability that two particles colliding (with momenta p_1, p_2) result into two other particles (with momenta p_3, p_4).



$$= \quad A(g, s, t, u)$$

- $A(g, s, t, u)$ depends on many things:
  - Which particles you are scattering (their masses, charges, etc)
  - The parameters of your theory g.
  - The momenta of the particles being scattered:

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2$$

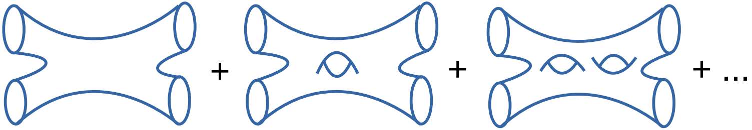
Scattering amplitudes

Why scattering amplitudes?

- They allow to test the predictions of our theory.
- They can teach us much about its structures/symmetries
  - On-shell Methods.
  - Color/kinematic duality and double copy.
  - Positive geometry.
  - $\cdots$

We would like to study string theory scattering amplitudes

- The computation organises in a genus expansion



$$A^{(genus\ 0)}(\alpha', s, t, u) + g_s^2 A^{(genus\ 1)}(\alpha', s, t, u) + g_s^4 A^{(genus\ 2)}(\alpha', s, t, u) + \cdots$$

At tree level and in flat space: Virasoro-Shapiro amplitude

$$A^{(genus\ 0)}(\alpha', s, t, u) = \frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})}$$

$$A^{(genus\ 0)}(\alpha', s, t, u) \sim \frac{1}{\underbrace{s\,t\,u}_{\text{sugra}}} + \underbrace{\alpha'^3 + \alpha'^5(s^2+t^2+u^2) + \cdots}_{\text{stringy corrections}}$$

- It mimics the structure of the low energy effective action

$$A^{(genus\ 0)}(\alpha', s, t, u) \sim \frac{1}{s\,t\,u} + \alpha'^3 + \alpha'^5(s^2+t^2+u^2) + \cdots$$

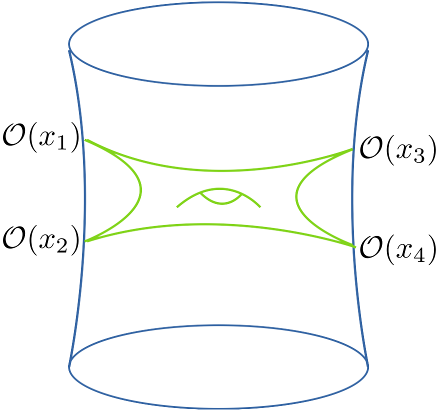
$$\mathcal{L}_{ST} = \mathcal{L}_{sugra} + \alpha'^3\mathcal{R}^4 + \alpha'^5\partial^4\mathcal{R}^4 + \cdots$$

- But being an on-shell quantity is easier to study.

What can we say about string theory amplitudes on *AdS*?

AdS/CFT

String amplitudes on *AdS* $\leftrightarrow$ Correlators of local operators in the CFT at the boundary.



$$\mathcal{A}(g_s, \alpha', s, t, u) \leftrightarrow \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$$

AdS/CFT

String theory on $AdS_5 \times S^5 \leftrightarrow$ 4d $\mathcal{N} = 4$ SYM
(g_s, R) (g_{YM}, N)

$$g_s \approx \frac{1}{N}, \qquad \frac{R^2}{\alpha'} = \sqrt{g_{YM}^2 N} \equiv \sqrt{\lambda}$$

String amplitudes on $AdS_5 \times S^5$	Correlators in $\mathcal{N} = 4$ SYM
Genus expansion	$1/N$ expansion
Stringy corrections to sugra	$1/\lambda$ corrections
Graviton on AdS	$\mathcal{O}_2$: protected scalar of dim. 2 in the stress-tensor multiplet
KK-modes on the S^5	$\mathcal{O}_k$: tower of scalar operators of dim. k

Compute $\langle \mathcal{O}_{k_1}(x_1)\mathcal{O}_{k_2}(x_2)\mathcal{O}_{k_3}(x_3)\mathcal{O}_{k_4}(x_4)\rangle$ in a $1/N$ and $1/\lambda$ expansion.

4d $\mathcal{N} = 4$ SYM - Kinematics

The symmetry

$$PSU(2,2|4) \supset \underbrace{SO(2,4)}_{\text{conformal symmetry}} \oplus \underbrace{SO(6)}_{\text{R-symmetry}}$$

The operators

$$\underbrace{\mathcal{O}_{I_1 \cdots I_k}(x)}_{\text{symmetric traceless of } SO(6)} \quad \rightarrow \quad \mathcal{O}_k(x,y) = \mathcal{O}_{I_1 \cdots I_k}(x) y^{I_1} \cdots y^{I_k}, \quad y^2 = 0$$

The observable

$$\langle \mathcal{O}_{k_1}(x_1,y_1) \mathcal{O}_{k_2}(x_2,y_2) \mathcal{O}_{k_3}(x_3,y_3) \mathcal{O}_{k_4}(x_4,y_4) \rangle$$

4d $\mathcal{N} = 4$ SYM - Kinematics

Bosonic symmetry

$$\langle \mathcal{O}_k(x_1, y_1) \cdots \mathcal{O}_k(x_4, y_4) \rangle = \left(\frac{y_{12} y_{34}}{x_{12}^2 x_{34}^2} \right)^k \underbrace{\mathcal{G}(U, V; \sigma, \tau)}_{\text{degree } k \text{ pol. in } \sigma, \tau}$$

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \sigma = \frac{y_1 \cdot y_3 y_2 \cdot y_4}{y_1 \cdot y_2 y_3 \cdot y_4}, \tau = \frac{y_1 \cdot y_4 y_2 \cdot y_3}{y_1 \cdot y_2 y_3 \cdot y_4}$$

Super-conformal Ward identities

$$(z\partial_z - \alpha \partial_\alpha) \mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha})|_{\alpha = 1/z} = 0$$
$$(z\partial_z - \bar{\alpha} \partial_{\bar{\alpha}}) \mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha})|_{\bar{\alpha} = 1/z} = 0$$

$$U = z\bar{z}, V = (1-z)(1-\bar{z}), \quad \sigma = \alpha\bar{\alpha}, \tau = (1-\alpha)(1-\bar{\alpha})$$

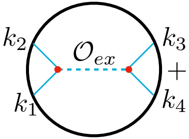
$1/N$ expansion

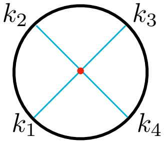
- Genus-0 amplitudes $\to$ leading non-trivial term in a $1/N$ expansion.
- Dropping $1/\lambda$ corrections gives the supergravity approximation.

$$\mathcal{G}(U,V;\sigma,\tau) = \underbrace{\mathcal{G}_{disc}(U,V;\sigma,\tau)}_{disconnected} + \frac{1}{N^2}\left(\boxed{\mathcal{G}^{(sugra)}(U,V;\sigma,\tau)} + \cdots\right) + \cdots$$

Standard recipe

1. Perform a KK reduction of the 10D Sugra effective action on S^5.
2. This give us an effective action on AdS_5, from where we can read off cubic and quartic vertices.
3. Write down & compute all exchange and contact Witten diagrams.

$\mathcal{G}^{(sugra)} = \sum_{\text{exch.}}$ 

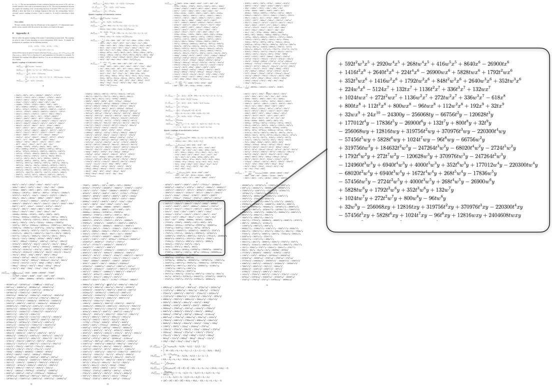
$+$ crossed $+$ 

Standard recipe

Several problems

- Effective action extremely hard to compute - 15 pp!

- Huge amount of Witten diagrams!
- Which are very hard to compute!
- For $AdS_5 \times S^5$ some heroic progress. **[Arutyunov, Frolov, Dolan, Osborn, Sokatchev, ...]**
- Hopeless for $AdS_7 \times S^4$ and $AdS_4 \times S^7$!



Issue: No clear understanding of the underlying organising principles.

We need a better way!

The right language: Mellin space

$\mathcal{G}(U,V;\sigma,\tau) \to \mathcal{M}(s,t,u;\sigma,\tau) \equiv \mathcal{M}(s,t;\sigma,\tau)$, with $s+t+u=4k$.

$$\boxed{\mathcal{G}(U,V;\sigma,\tau) = \int_{-i\infty}^{i\infty} dsdtU^sV^t \underbrace{\Gamma_{\{k_i\}}(s,t,u)}_{\text{A prefactor}} \quad \underbrace{\mathcal{M}(s,t;\sigma,\tau)}_{\text{string amplitude in } AdS_5\times S^5}}$$

$\mathcal{M}(s,t;\sigma,\tau)$ is a meromorphic function with very nice properties:

1. Crossing symmetry. $\checkmark$
2. Exchanged operators lead to simple poles:

$$\mathcal{M}_{exch}(s,t) = \sum_{m=0}^{\infty} \lambda^2_{\Delta,\ell} \frac{Q_{\ell,m}(u,t)}{s-(\Delta-\ell)-2m} + \text{regular} \quad \checkmark$$

3. Superconformal Ward identities:

$$(\text{Shift operator}) \circ \mathcal{M}(s,t;\sigma,\tau) = 0 \quad \checkmark$$

Rather than the effective action use 1+2+3!

MRV amplitudes

One can produce results case by case, and its ok for $AdS_5 \times S^5$ **[Rastelli, Zhou]**, but its hard to generalise to other cases...

We need a new idea!

Maximally R-symmetry violating amplitudes

- Each operator depends on a 4d point x and a 6d null vector y:

$$\langle \mathcal{O}_k(x_1, y_1)\mathcal{O}_k(x_2, y_2)\mathcal{O}_k(x_3, y_3)\mathcal{O}_k(x_4, y_4)\rangle$$

Point in $\mathbb{R}^{1,3}$ Null 6d vector

- Choose a configuration where y_1, y_3 are aligned $\rightarrow \sigma = 0, \tau = 1$.

$$\boxed{MRV(s,t) = \mathcal{M}(s,t;0,1)}$$

This suppresses all sugra exchanges in the u-channel (as $y_1 \cdot y_3 = 0$) and the amplitude simplifies drastically!

MRV amplitudes

Stress tensor multiplet four-point function in $AdS_5 \times S^5$

```
In[∘]:= M2 =
  -((192 - 80 s + 8 s^2 - 224 t + 76 s t - 6 s^2 t + 92 t^2 - 22 s t^2 + s^2 t^2 - 16 t^3 + 2 s t^3 + t^4 -
      64 σ + 128 s σ - 16 s^2 σ + 48 t σ - 112 s t σ + 12 s^2 t σ - 8 t^2 σ + 28 s t^2 σ - 2 s^2 t^2 σ -
      2 s t^3 σ + 64 σ^2 - 48 s σ^2 + 8 s^2 σ^2 - 48 t σ^2 + 36 s t σ^2 - 6 s^2 t σ^2 + 8 t^2 σ^2 - 6 s t^2 σ^2 +
      s^2 t^2 σ^2 - 192 τ + 80 s τ - 8 s^2 τ + 80 t τ + 32 s t τ - 20 s^2 t τ + 2 s^3 t τ - 8 t^2 τ -
      20 s t^2 τ + 4 s^2 t^2 τ + 2 s t^3 τ - 64 σ τ + 48 s σ τ - 8 s^2 σ τ + 128 t σ τ - 112 s t σ τ +
      28 s^2 t σ τ - 2 s^3 t σ τ - 16 t^2 σ τ + 12 s t^2 σ τ - 2 s^2 t^2 σ τ + 192 τ^2 - 224 s τ^2 + 92 s^2 τ^2 -
      16 s^3 τ^2 + s^4 τ^2 - 80 t τ^2 + 76 s t τ^2 - 22 s^2 t τ^2 + 2 s^3 t τ^2 + 8 t^2 τ^2 - 6 s t^2 τ^2 + s^2 t^2 τ^2) /
    (2 (-2 + s) (-2 + t) (-6 + s + t)));
```

Stress tensor multiplet four-point function in $AdS_5 \times S^5$

```
In[∘]:= M2 =
  -((192 - 80 s + 8 s^2 - 224 t + 76 s t - 6 s^2 t + 92 t^2 - 22 s t^2 + s^2 t^2 - 16 t^3 + 2 s t^3 + t^4 -
      64 σ + 128 s σ - 16 s^2 σ + 48 t σ - 112 s t σ + 12 s^2 t σ - 8 t^2 σ + 28 s t^2 σ - 2 s^2 t^2 σ -
      2 s t^3 σ + 64 σ^2 - 48 s σ^2 + 8 s^2 σ^2 - 48 t σ^2 + 36 s t σ^2 - 6 s^2 t σ^2 + 8 t^2 σ^2 - 6 s t^2 σ^2 +
      s^2 t^2 σ^2 - 192 τ + 80 s τ - 8 s^2 τ + 80 t τ + 32 s t τ - 20 s^2 t τ + 2 s^3 t τ - 8 t^2 τ -
      20 s t^2 τ + 4 s^2 t^2 τ + 2 s t^3 τ - 64 σ τ + 48 s σ τ - 8 s^2 σ τ + 128 t σ τ - 112 s t σ τ +
      28 s^2 t σ τ - 2 s^3 t σ τ - 16 t^2 σ τ + 12 s t^2 σ τ - 2 s^2 t^2 σ τ + 192 τ^2 - 224 s τ^2 + 92 s^2 τ^2 -
      16 s^3 τ^2 + s^4 τ^2 - 80 t τ^2 + 76 s t τ^2 - 22 s^2 t τ^2 + 2 s^3 t τ^2 + 8 t^2 τ^2 - 6 s t^2 τ^2 + s^2 t^2 τ^2) /
    (2 (-2 + s) (-2 + t) (-6 + s + t)));
```

Much simpler in the MRV limit!

$$MRV(s,t) = (u-4)(u-6)\left(\frac{1}{s-2} + \frac{1}{t-2}\right)$$

MRV amplitudes

$$MRV(s,t) = \underbrace{(u-4)(u-6)}_{\text{double zero}} \left(\underbrace{\frac{1}{s-2} + \frac{1}{t-2}}_{\text{no poles in the } u\text{-channel}} \right)$$

- No poles in u and always a double-zero.
- Highly non-trivial at the level of exchange Witten diagrams. Powerful relations among cubic couplings!
- We can write down the general MRV amplitude for all k_1, k_2, k_3, k_4!
- $R-$symmetry can be used to restore the full σ, τ dependence!

$$MRV(s,t) \xrightarrow{\text{R-symmetry}} \mathcal{M}(s,t,\sigma,\tau)$$

Final (supergravity) solution

- Representation where contact terms are actually absent!

$$\mathcal{M}^{(sugra)}(s,t;\sigma,\tau) = \sum_{i,j} \sigma^i \tau^j \left(\mathcal{M}_s^{i,j}(s,t) + \mathcal{M}_t^{i,j}(s,t) + \mathcal{M}_u^{i,j}(s,t) \right)$$

$$\mathcal{M}_s^{i,j}(s,t) = \sum_h \frac{R_h^{i,j}(t,u)}{s-h}$$

explicitly given!

- Compact and explicit expression for all supergravity amplitudes!
- For $AdS_5 \times S^5$ this proves the results by Rastelli and Zhou.

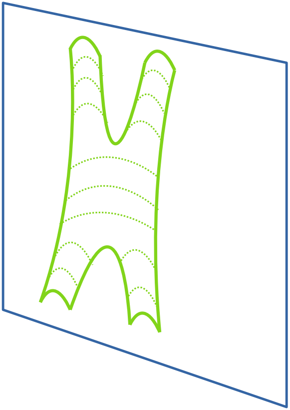
But even more...

- It actually works for all maximally susy theories, holographically dual to $AdS_7 \times S^4$, $AdS_5 \times S^5$ and $AdS_4 \times S^7$!

Gluon scattering in AdS

Let's now scatter gluons!

Gluons $\rightarrow$ open strings $\rightarrow$ add branes



- Add M $D7$ branes wrapping AdS_5 and $S^3 \subset S^5$ ($M \ll N$)

Gluon scattering in AdS

- The presence of $D7-$branes breaks the symmetry

$$\begin{aligned} \mathcal{N}=4 \text{ SYM} &\rightarrow \mathcal{N}=2 \text{ SYM with flavours} \\ SO(6) &\rightarrow SO(4)\times SO(2) = SU(2)_L \times \underbrace{SU(2)_R\times U(1)}_{\text{4d } \mathcal{N}=2 \text{ R-symmetry}} \end{aligned}$$

- The global flavour symmetry gives rise to a spin-1 current multiplet

Gluon amplitudes on AdS	Correlators in $\mathcal{N}=2$ SYM w/flavours
Gluon on AdS	$\mathcal{O}_2$: protected scalar of dim. 2 in the current multiplet
KK-modes on the S^3	$\mathcal{O}_k$: tower of scalar operators of dim. k

The operators

$$\overset{\text{Adj. of } SU(M)}{\downarrow}$$

$$\mathcal{O}^{I}_{\underbrace{\alpha_1\cdots\alpha_k}_{\frac{k}{2}\text{ of } SU(2)_R};\underbrace{\bar{\alpha}_1\cdots\bar{\alpha}_{k-2}}_{\frac{k-2}{2}\text{ of } SU(2)_L}}(x)$$

$$\mathcal{O}^{I}_{k}(x,v,\bar{v}) = \mathcal{O}^{I}_{\alpha_1\cdots\alpha_k;\bar{\alpha}_1\cdots\bar{\alpha}_{k-2}}(x)v^{\alpha_1}\cdots v^{\alpha_k}\bar{v}^{\bar{\alpha}_1}\cdots\bar{v}^{\bar{\alpha}_{k-2}}$$

$SU(2)$ spinors

The observable

$$\langle\mathcal{O}^{I_1}_{k}(x_1,v_1,\bar{v}_1)\cdots\mathcal{O}^{I_4}_{k}(x_4,v_4,\bar{v}_4)\rangle = \text{pref.}\times\underbrace{\mathcal{G}^{I_1I_2I_3I_4}(U,V;\alpha,\beta)}_{\text{degree } k \text{ in } \alpha,\ k-2 \text{ in } \beta}$$

$$\alpha = \frac{v_1\cdot v_3\, v_2\cdot v_4}{v_1\cdot v_2\, v_3\cdot v_4},\quad \beta = \frac{\bar{v}_1\cdot\bar{v}_3\,\bar{v}_2\cdot\bar{v}_4}{\bar{v}_1\cdot\bar{v}_2\,\bar{v}_3\cdot\bar{v}_4}$$

Gluon scattering in AdS

Super-conformal Ward identities

$$\left(z\partial_z - \alpha\partial_\alpha\right)\mathcal{G}^{I_1 I_2 I_3 I_4}(z,\bar{z};\alpha,\beta)\Big|_{\alpha=1/z} = 0$$

- Naively half as powerful, as we don't have $\alpha \leftrightarrow \beta$...
- But the independent flavour structures give rise to more constraints:

Super-conformal Ward identities hold for all $\mathcal{R} \in Adj \times Adj$

- Ends up being as powerful!

Gluon scattering in AdS

- We are interested in the tree-level amplitude

$$\mathcal{M}^{I_1 I_2 I_3 I_4}_{(tree)}(s,t;\alpha,\beta)$$

- To leading order in $1/N$ the exchange of gravitons is suppressed.

The procedure

- Consider the MRV limit (v_1, v_3 aligned)

$$MRV(s,t,\beta) = \mathcal{M}(s,t;\alpha=0,\beta)$$

1. No poles in u.
2. Zero at $u = 2k$.

- Again a powerful set of constraints which fully fix the MRV answer.
- Again use R-symmetry to get the full amplitude!

$$MRV(s,t,\beta) \xrightarrow{\text{R-symmetry}} \mathcal{M}(s,t,\alpha,\beta)$$

Gluon scattering in AdS

Gluon amplitudes on AdS

$$c_s = f^{I_1 I_2 J} f^{J I_3 I_4}$$

$$\mathcal{M}^{I_1 I_2 I_3 I_4}_{(tree)}(s,t;\alpha,\beta) = c_s \mathcal{M}_s(s,t;\alpha,\beta) + c_t \mathcal{M}_t(s,t;\alpha,\beta) + c_u \mathcal{M}_u(s,t;\alpha,\beta)$$

$$\sum_h \frac{R_h(t,u;\alpha,\beta)}{s-h}$$

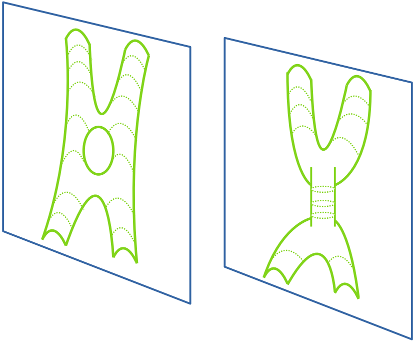
explicitly given!

- All colour dependence through c_s, c_t, c_u (as in flat space).
- Representation where contact terms are absent.
- The same method works for a variety of theories!
  - 4d $\mathcal{N}=2$ from $D3-$branes near F-theory singularities ✓
  - 5d Seiberg exceptional theories ✓
  - 6d E-string theory ✓
  - Probe branes in 3d and 4d (the case discussed here) ✓

Conclusions

We now have all tree level amplitudes, in a variety of maximally and half-maximally susy CFTs, in $3, 4, 5$ and 6 dimensions.

- Wealth of CFT-data encoded in these correlators. Can we feed this into other approaches?
- Loops and gravitational interactions?



- How much of the beautiful structure present in flat space is present in AdS? Consider higher point functions!