Renormalization Group Flows on Line Defects

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*based on arxiv:2108.01117 with **Zohar Komargodski** and **Avia Raviv-Moshe***

Strings, Fields and Holograms - 11 October 2021, Ascona

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Introduction and summary

Line defects in physics

Line *defects* appear in many physical systems. An incomplete list includes:

- 2d: boundaries and interfaces in systems at criticality (e.g. Kondo problem)

  Wilson 1975, Tsevelick Wiegmann 1985, Cardy 1989, Affleck 1995…

- 3d: symmetry defects, magnetic impurities in lattice systems,…

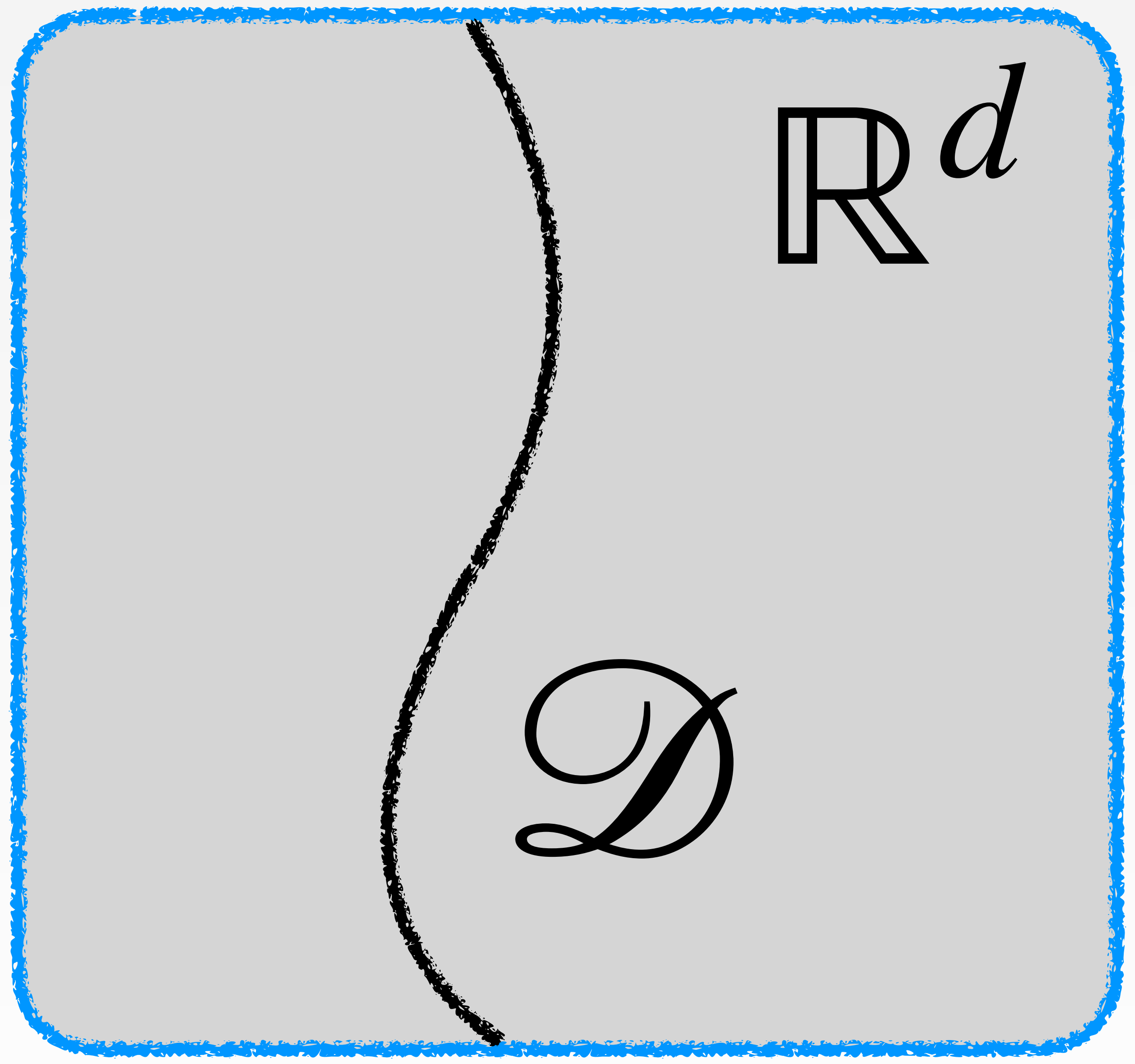
  Sachdev Buragohain Vojta 1999, Billó Caselle Gaiotto Gliozzi Meineri Pellegrini 2013…

- 4d: Wilson and 't Hooft lines

  Wilson 1974, 't Hooft 1978, Kapustin 2005…

Extended operators as defect QFTs

2 equivalent viewpoints on defects in Quantum Field Theory (QFT):

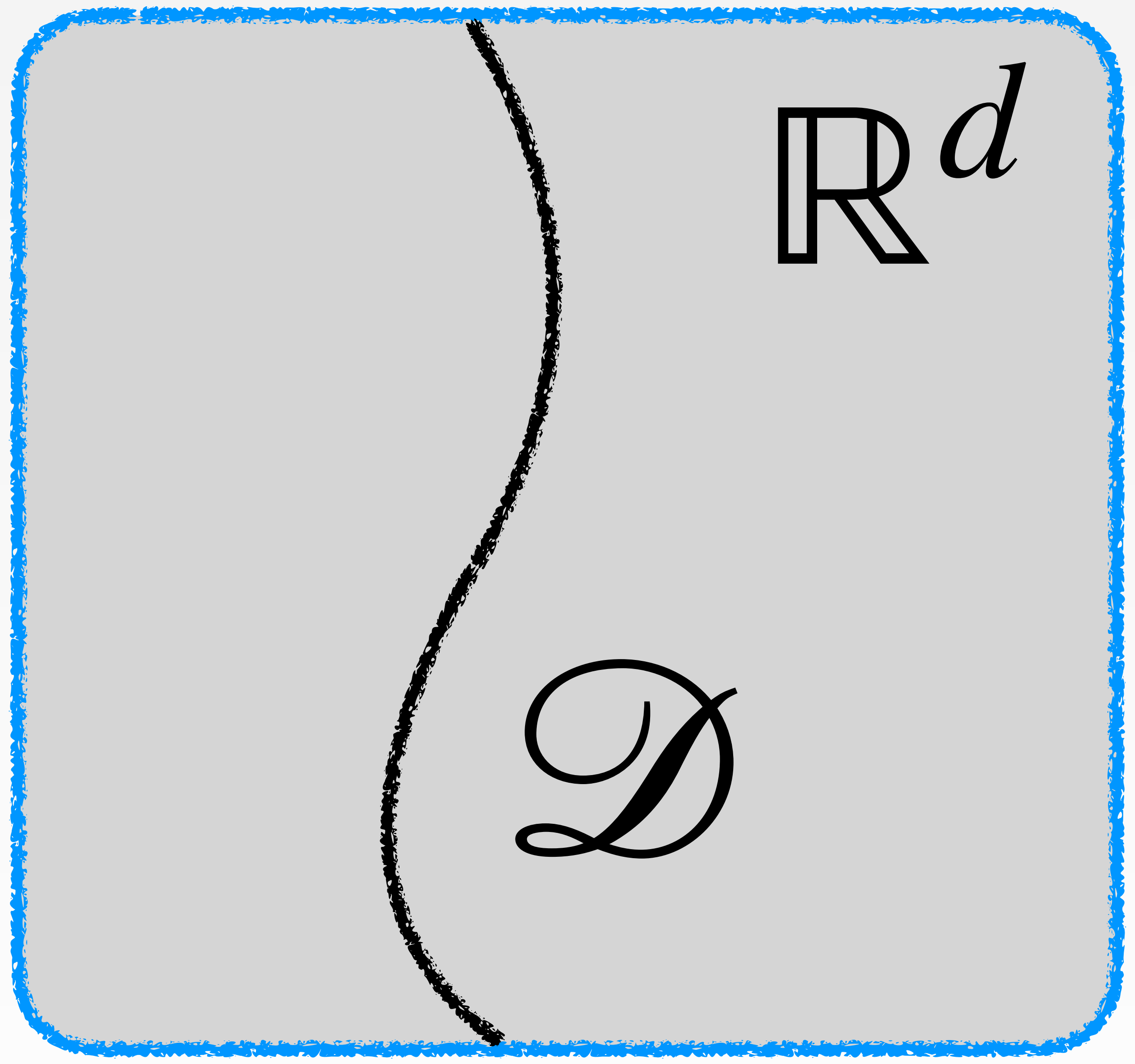


Extended operators as defect QFTs

2 equivalent viewpoints on defects in Quantum Field Theory (QFT):

- Defects are extended operators of the bulk QFT :

$$\mathcal{D} = e^{i \int_\Sigma \mathcal{L}(\phi)} \quad \leftrightarrow \quad \langle \mathcal{D} \ldots \rangle = \int_{\phi(\Sigma)=\phi_0} D\phi e^{iS[\phi]}$$



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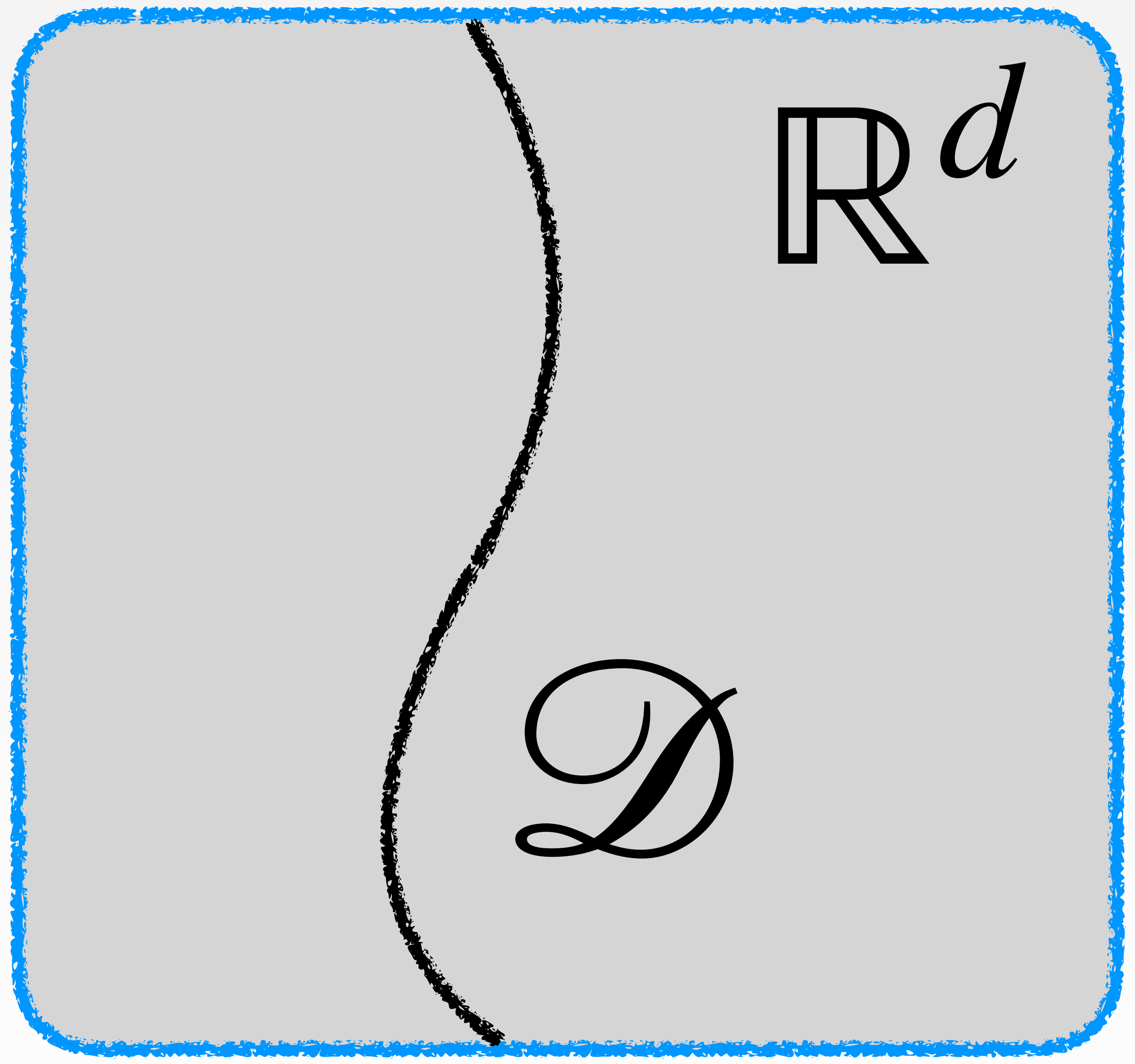
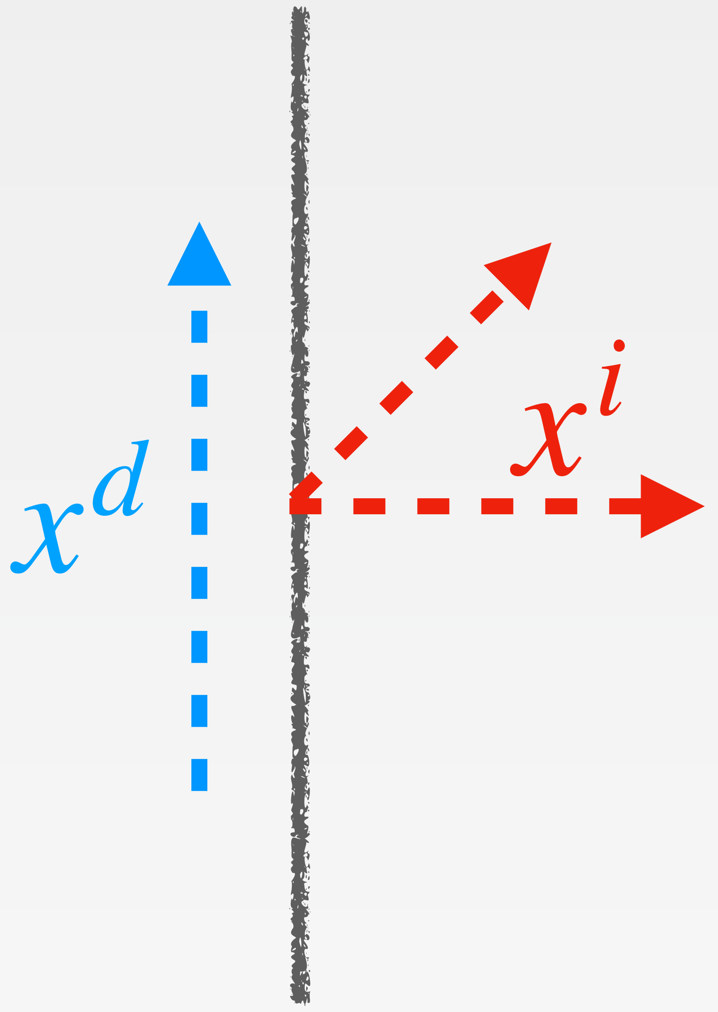


- Bulk+defect define a *Defect QFT* (DQFT):

$$\mathcal{L} = \mathcal{L}_{bulk} + \delta_{\mathcal{D}}^{d-1}\mathcal{L}_{defect}$$

Defects in conformal field theories (CFTs)



Defect CFTs (DCFTs) = DQFTs preserving the 1d conformal group when placed on straight or circular lines

$$SO(d+1,1) \supset \overbrace{SL(2,\mathbb{R})}^{\text{Conformal along } x^d} \times \overbrace{SO(d-1)}^{\text{Rotations of } x^i}$$

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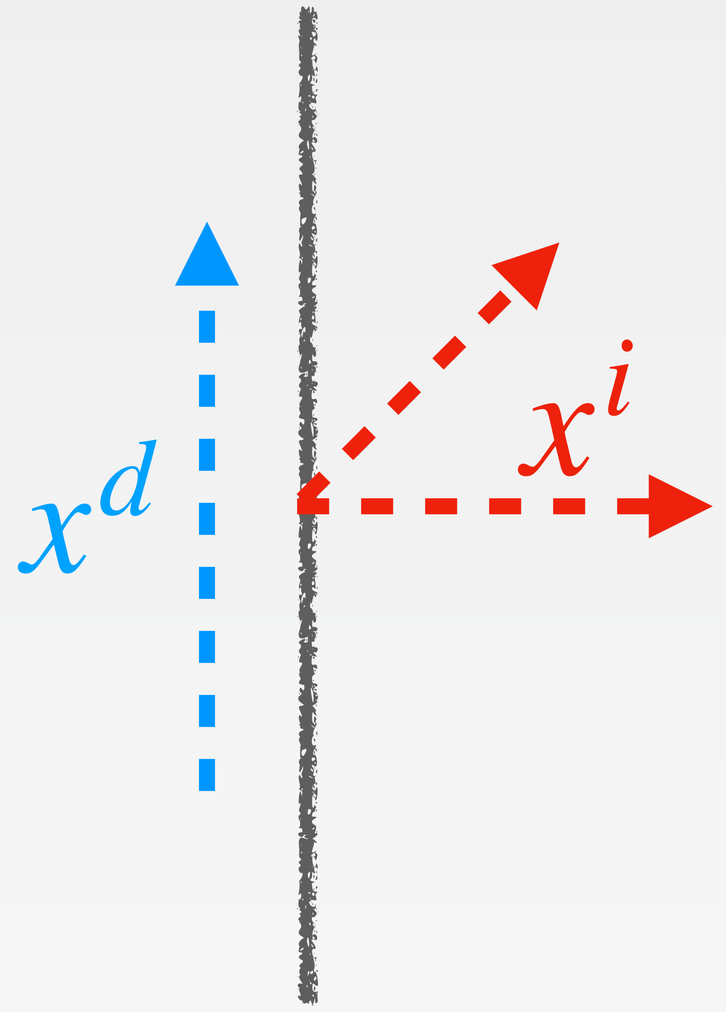

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Massive DQFTs flow under the *defect renormalization group* (DRG) between UV and IR DCFT fixed points

$$\mathit{DCFT}_{UV} \xrightarrow{\mathit{DRG}} \mathit{DCFT}_{IR}$$

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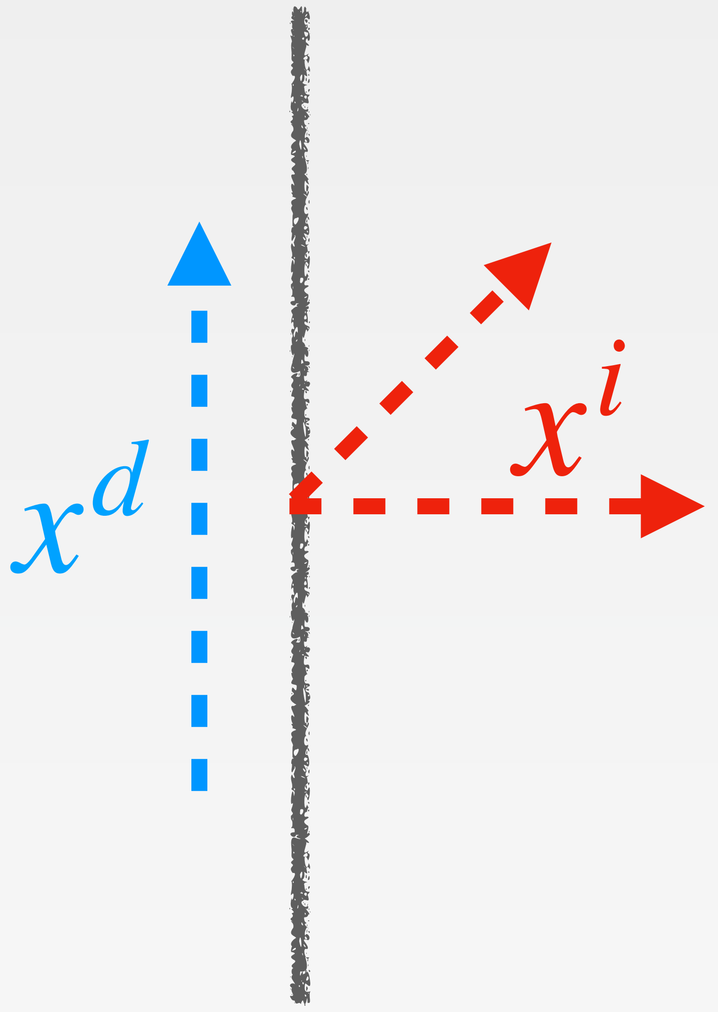

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Is the DRG flow irreversible?

Irreversibility of defect RG flows

Is the DRG reversible? Could we have DRG limit cycles?

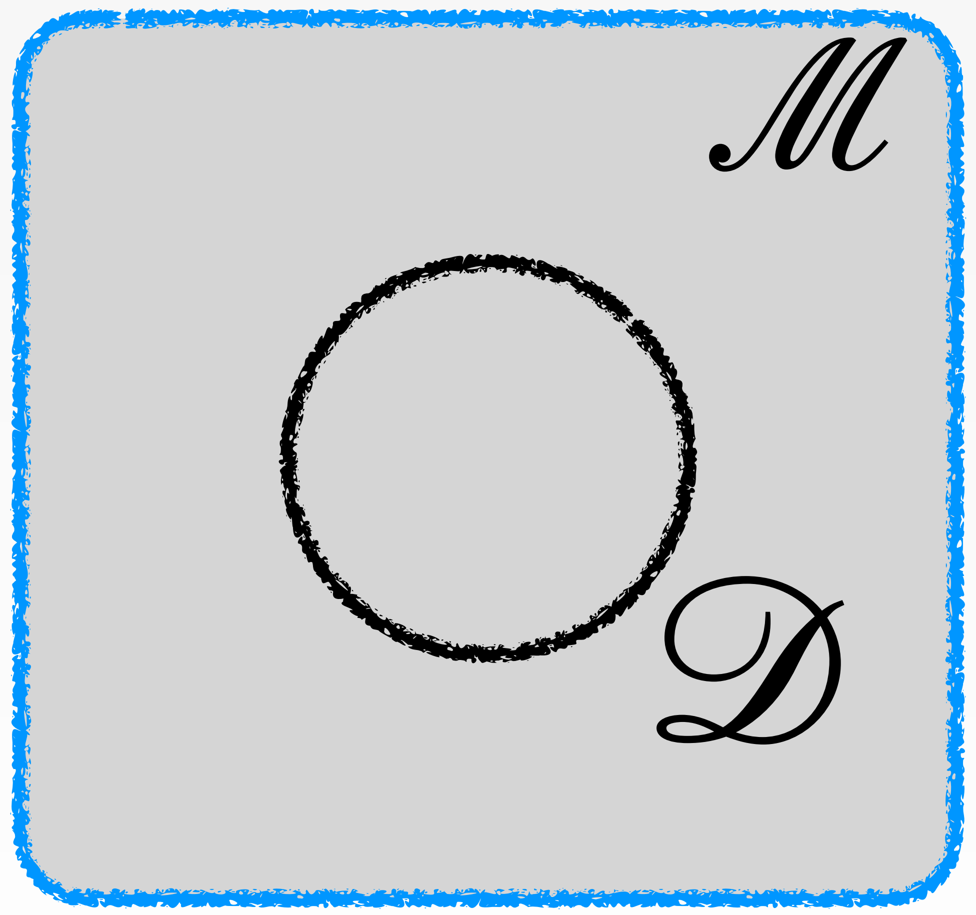
Irreversibility of defect RG flows

Is the DRG reversible? Could we have DRG limit cycles?

NO! We will prove that the (universal part of the) partition function of a circular defect defines a c-function for local unitary DCFTs:

$$\log g = \log Z_{DCFT} - \log Z_{CFT}$$

$$\boxed{g_{UV} > g_{IR}}$$



- Additionally, a monotonically decreasing entropy function exists
- Our construction generalizes the d=2 g-theorem to higher dimensions

Affleck Ludwig 1991, Friedan Konechny 2003, Casini Salazar-Landea Torroba 2016

Plan of the talk

1. Straight vs circular lines in DCFTs
2. DRG flows and spurion analysis
3. The defect entropy and the gradient formula
4. Examples
5. Conclusions and outlook

Straight vs circular lines in DCFTs

Local DCFTs

Locality of the defect is equivalent to the following Ward identity:

Bulk stress tensor

Displacement operator

$$\nabla_\mu T_b^{\mu\nu} = -\delta_{\mathcal{D}}^{d-1} n_i^\nu\, D^i$$

- <u>Important:</u> DCFTs do not support a defect stress tensor

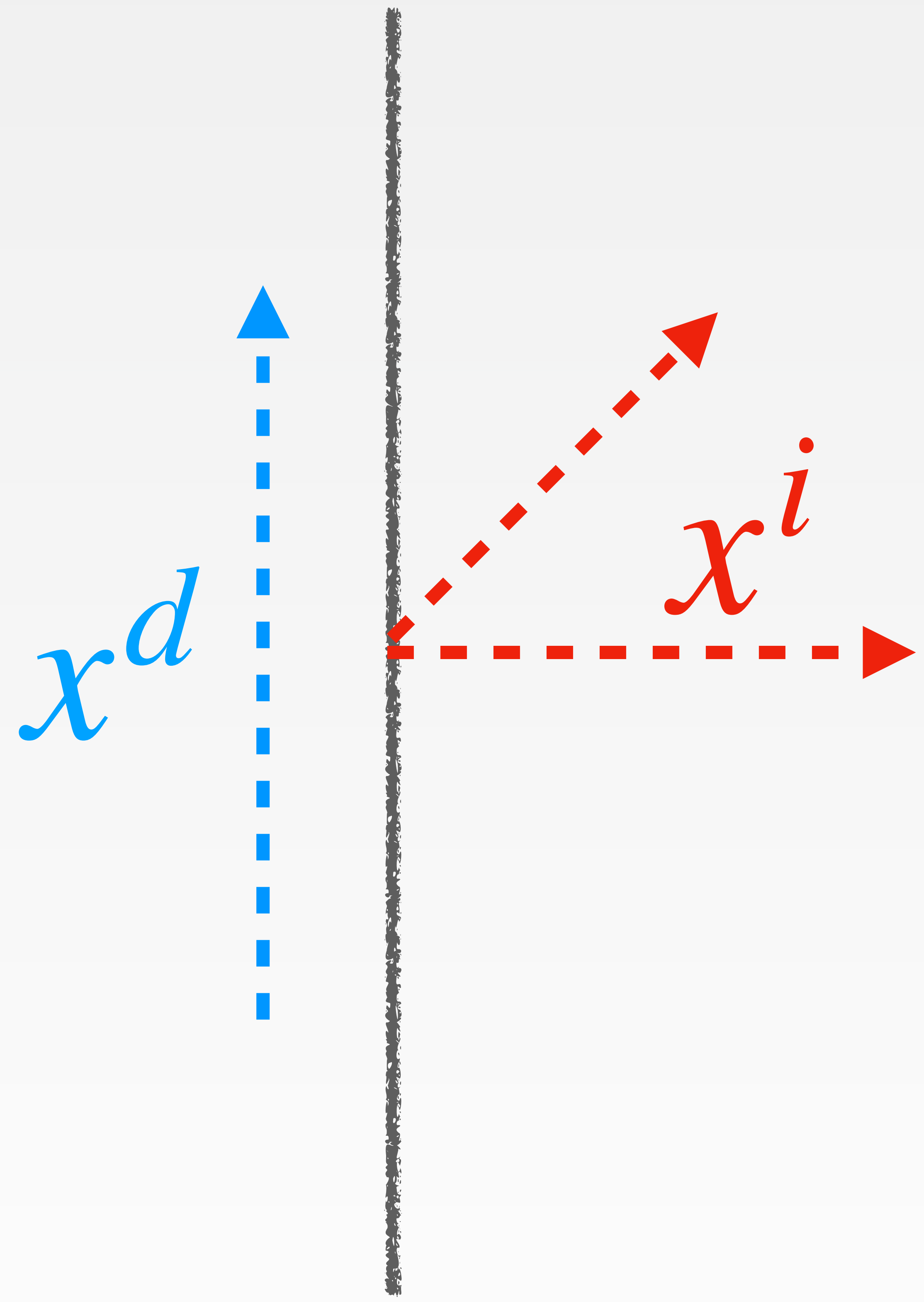
- Topological defects: $D^i = 0$

Straight = circular for DCFTs?

Conformal invariance fixes the 1pt function of the stress tensor up to a theory-dependent coefficient $h_{\mathscr{D}}$:

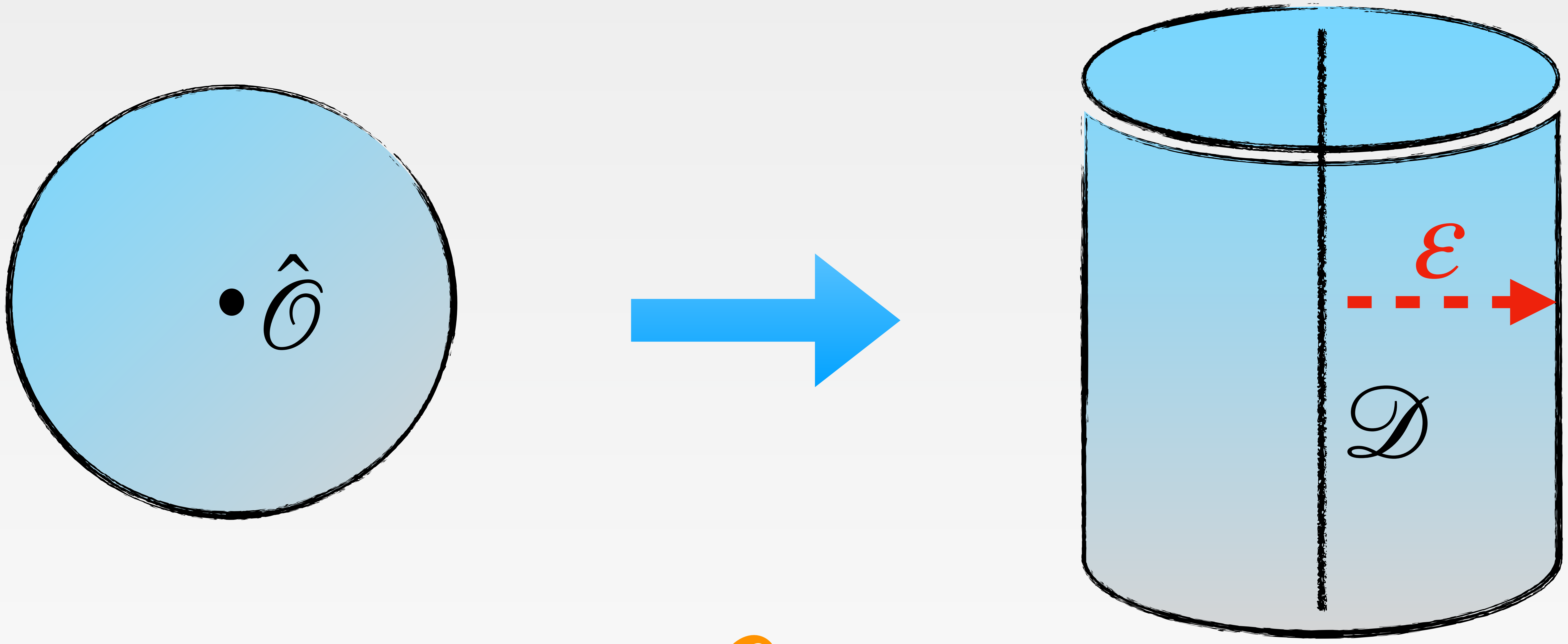
$$\langle T^{dd}(x)\rangle = h_{\mathcal{D}}\frac{d-2}{r^d}\,, \qquad \langle T^{id}(x)\rangle = 0\,,$$

$$\langle T^{ij}(x)\rangle = -h_{\mathcal{D}}\frac{(2\,\delta^{ij} - d\,x^i x^j/r^2)}{r^d}\,.$$



- In $d=2$ these imply $\langle T^{\mu\nu}(x)\rangle = 0$
- In $d>2$ the $x\to\infty$ limit depends on the distance r from the line

Consider wrapping the defect with the $SL(2,\mathbb{R})$ charges:



$$Q_\xi(\mathcal{D}) = \int_{r=\varepsilon} d^{d-1}\Sigma^\mu \langle T^b_{\mu\nu}\rangle \xi^\nu \overset{?}{=} 0$$

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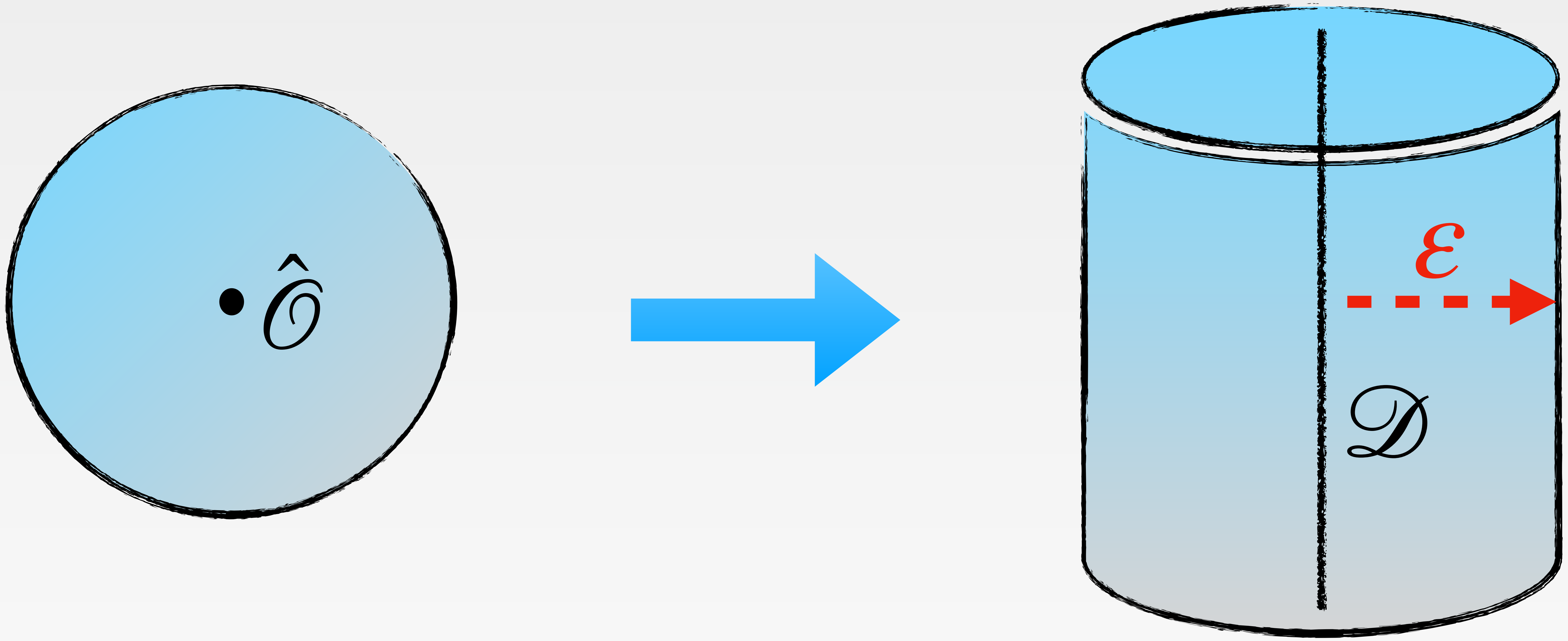


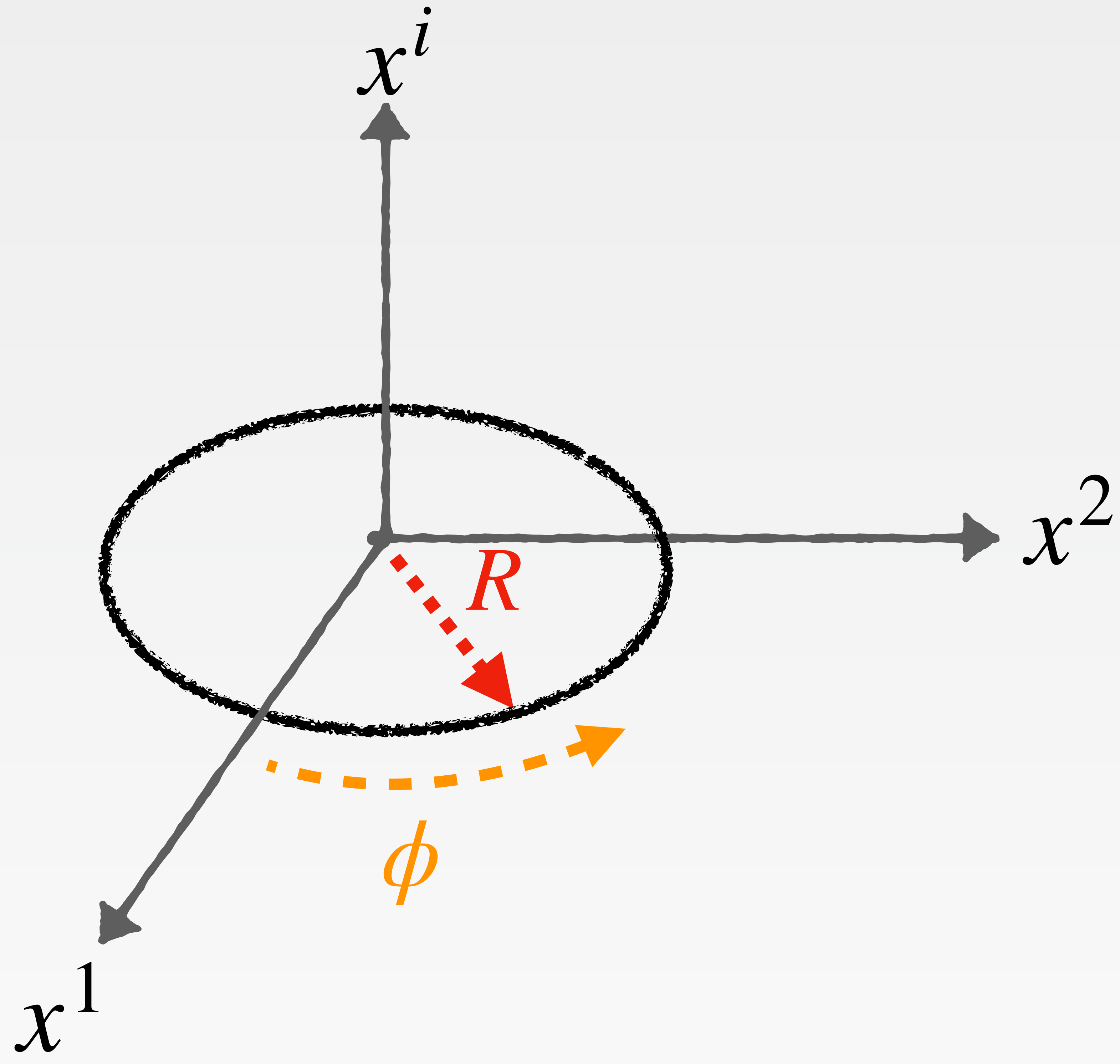
$$Q_\xi(\mathcal{D}) = \int_{r=\varepsilon} d^{d-1}\Sigma^\mu \langle T^b_{\mu\nu}\rangle \xi^\nu \neq 0 \quad \xrightarrow{\text{Dilations}} \quad Q_\xi(\mathcal{D}) \sim \int dx^d \times \frac{1}{\varepsilon}$$

Kapustin 2005

Presumably related to the disagreement $Z_{straight} \neq Z_{circle}$

Erickson Semenoff Zarembo 2000, Drukker Gross 2001

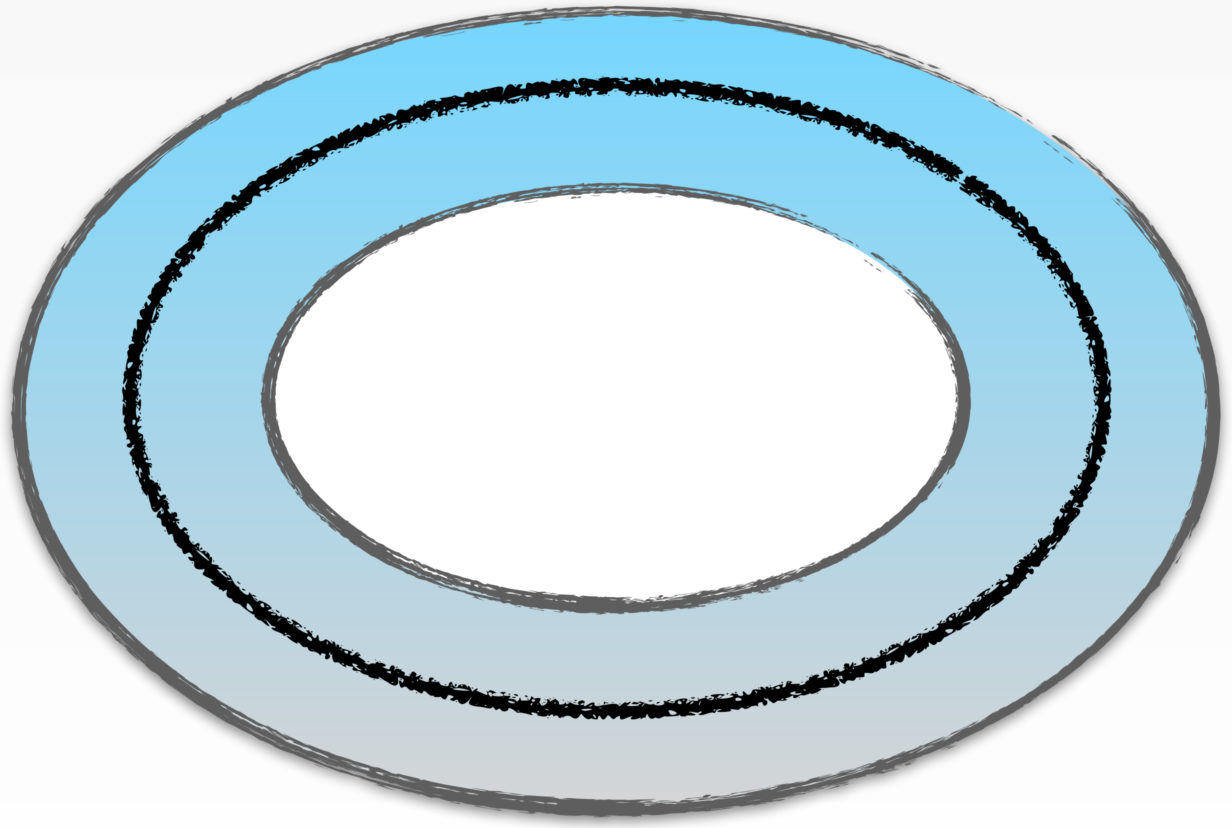
No issues of this sort arise for circular defects. The $SL(2,\mathbb{R})$ Killing vectors are



$$\begin{cases} \xi^\mu_{(1)} = \frac{1}{2}\left[\delta^\mu_1\left(R + x^2/R\right) - 2x^\mu x_1/R\right] & \xrightarrow{\mathcal{D}} -\sin\phi\,, \\ \xi^\mu_{(2)} = \frac{1}{2}\left[\delta^\mu_2\left(R + x^2/R\right) - 2x^\mu x_2/R\right] & \xrightarrow{\mathcal{D}} \cos\phi\,, \\ \xi^\mu_{(\phi)} = \delta^\mu_1 x^2 - \delta^\mu_2 x^1 & \xrightarrow{\mathcal{D}} -1 \end{cases}$$

$SL(2,\mathbb{R})$ charges manifestly vanish in this geometry:

$$Q_\xi(\mathcal{D}) = 0$$



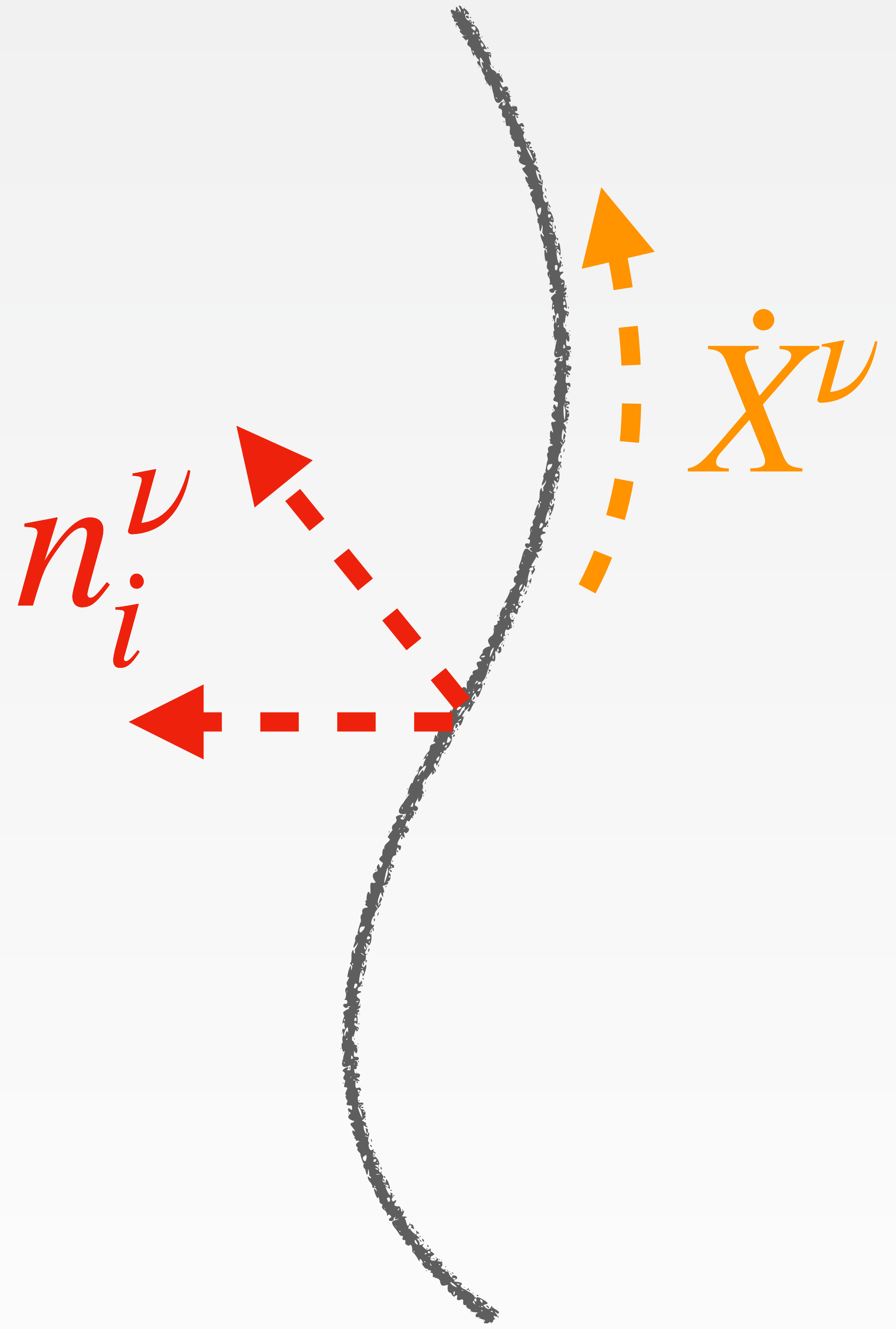
DRG flows and spurion analysis

Defect renormalization group

Relevant perturbations ($\Delta_{\mathcal{O}} < 1$) trigger a DRG:

$$S_{DCFT} \to S_{DCFT} + M_0^{1-\Delta_{\mathcal{O}}} \int_{\mathcal{D}} d\sigma \mathcal{O}(\sigma)$$

Broken scale invariance allows to localize energy on the defect:



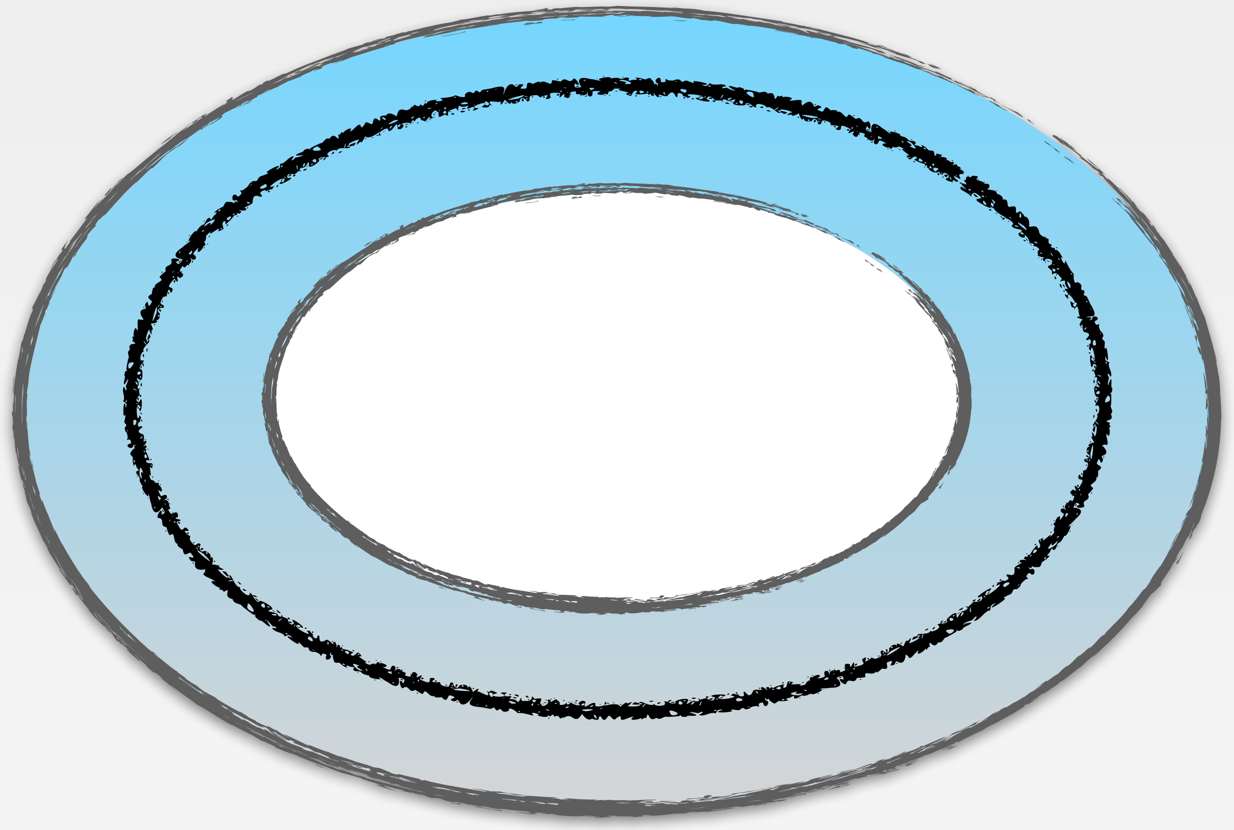
$$\nabla_\mu T_b^{\mu\nu} = -\delta_{\mathcal{D}}^{d-1} \dot{X}^\nu \dot{T}_D - \delta_{\mathcal{D}}^{d-1} n_i^\nu D^i$$

Defect Stress Tensor

- In perturbation theory $T_D = \beta_{\mathcal{O}} \mathcal{O}$

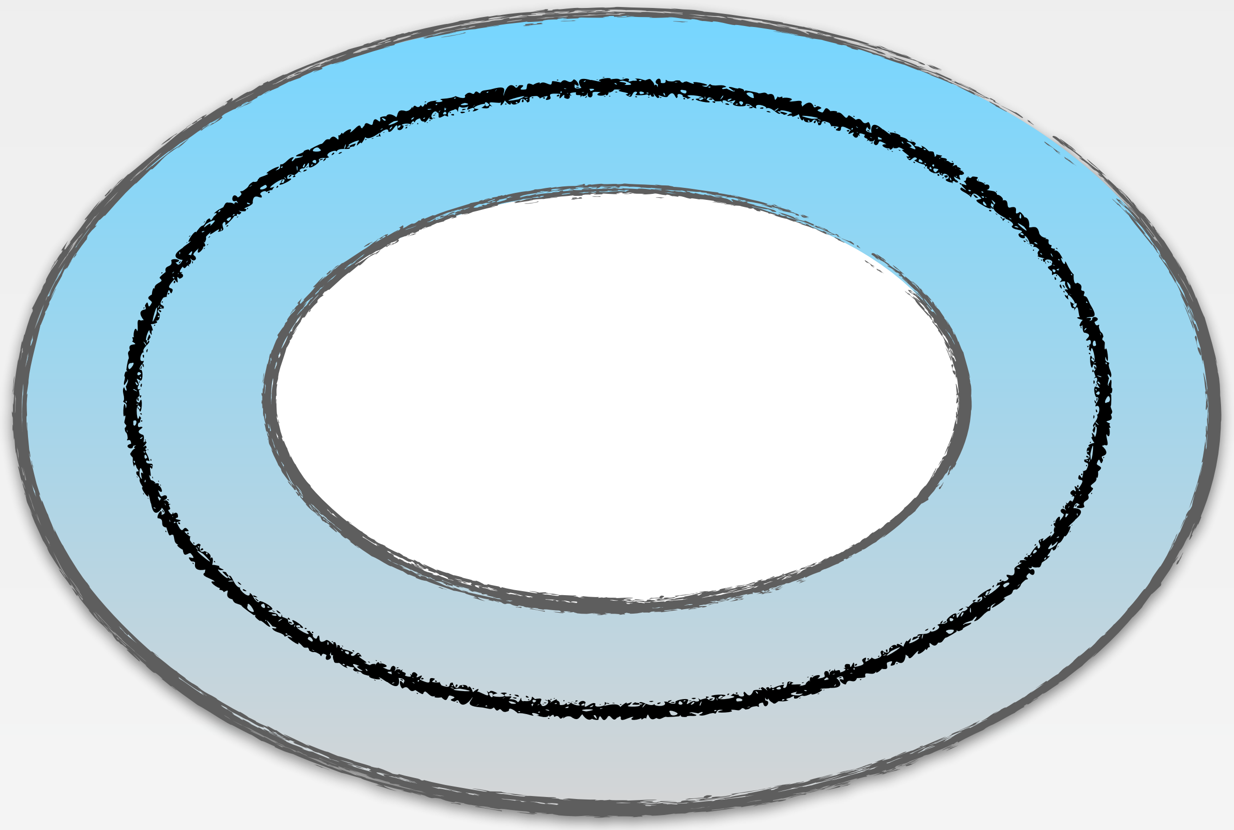
Spurion analysis and equivalent RGs

Conformal invariance of the vacuum still requires



$$\langle Q_\xi(\mathcal{D})\rangle = 0$$

Spurion analysis and equivalent RGs



Conformal invariance of the vacuum still requires

$$\langle Q_\xi(\mathcal{D})\rangle = 0$$

In general $Q_\xi(\mathscr{D}) \neq 0$. We introduce a background dilaton field:

$$M(\sigma) = M_0 e^{\Phi(\sigma)}$$

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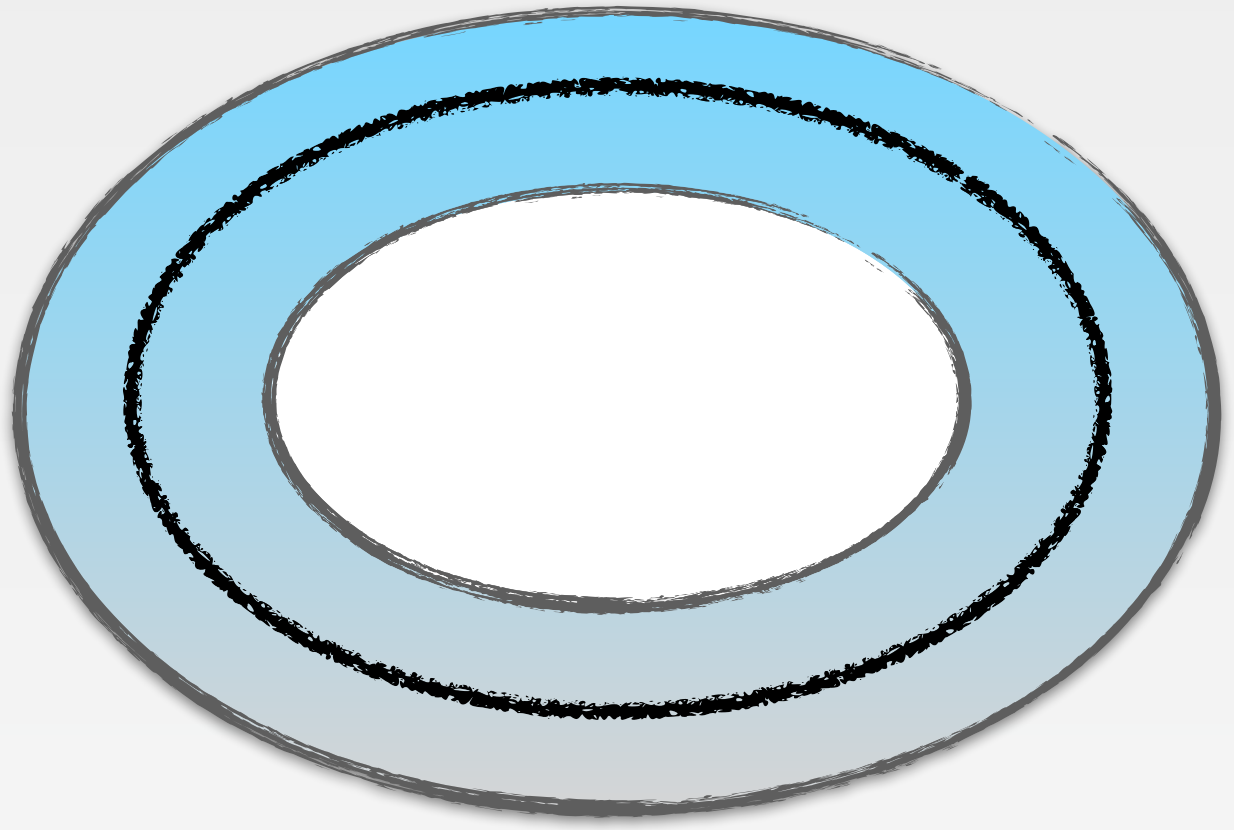

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$$M(\sigma) = M_0 e^{\Phi(\sigma)}$$

$SL(2,\mathbb{R})$ charges relate theories with different sources $\Phi(\sigma)$:

$$\Phi \sim \Phi + \alpha\left(\dot{\xi}_{\mathcal{D}} + \xi_{\mathcal{D}}\dot{\Phi}\right), \qquad \alpha \ll 1$$

$SL(2,\mathbb{R})$ CKV on the circle

Consider $\xi_{\mathscr{D}} = \sin\phi$:

$$\Phi \sim \Phi + \alpha\left(\cos\phi + \sin\phi\,\dot{\Phi}\right)\;, \qquad \alpha \ll 1$$

$$\boxed{\textit{Certain DRG flows triggered by different space dependent mass scales are equivalent}}$$

Consider $\xi_{\mathscr{D}} = \sin\phi$:

$$\Phi \sim \Phi + \alpha \left(\cos\phi + \sin\phi\, \dot{\Phi}\right) \,, \qquad \alpha \ll 1$$

Certain DRG flows triggered by different space dependent mass scales are equivalent

- This equivalence provides nontrivial identities expanding the partition function around $\Phi = 0$. In the following we will need one of them:

$$R \int_{\mathcal{D}} d\phi \langle T_D(\phi)\rangle = R^2 \int_{\mathcal{D}} d\phi_1 \int_{\mathcal{D}} d\phi_2 \langle T_D(\phi_1) T_D(\phi_2)\rangle \cos(\phi_1 - \phi_2)$$

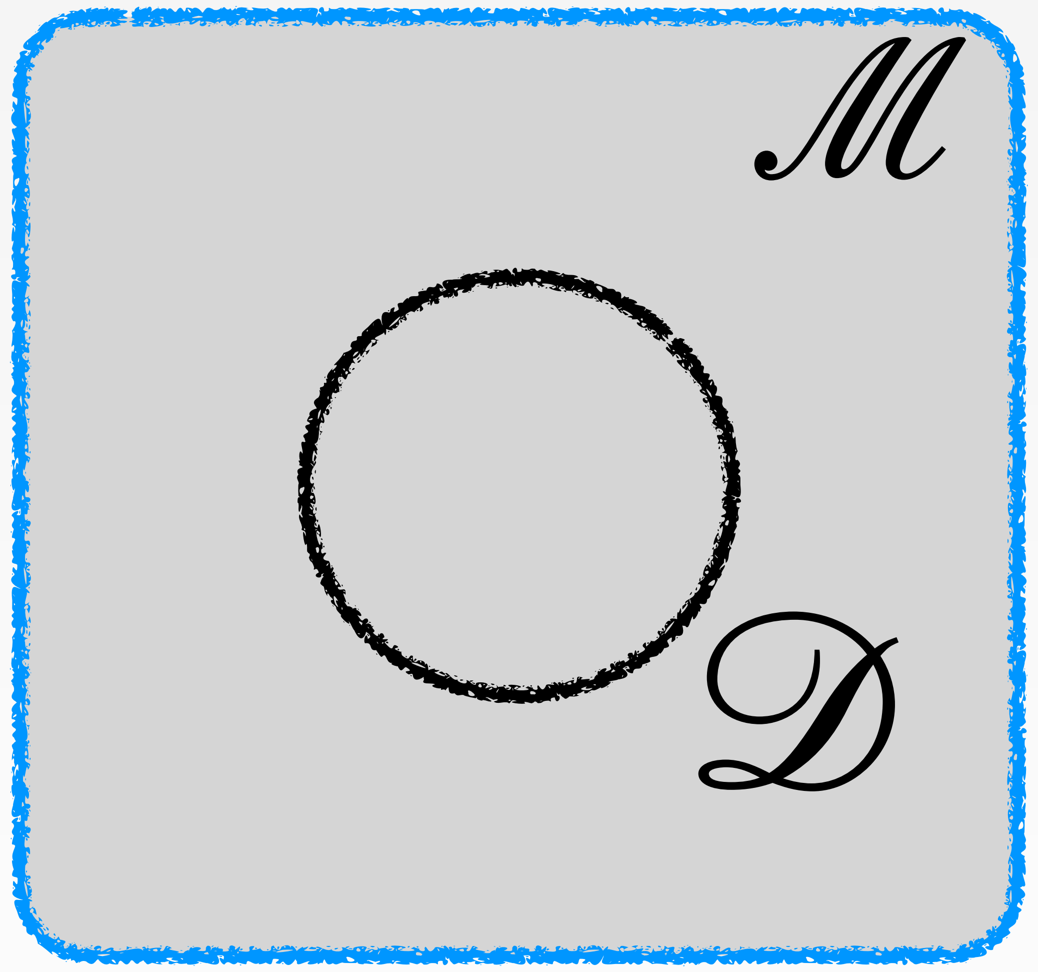
The defect entropy and the gradient formula

The g-function and the defect entropy

The circular defect contribution to the partition function defines the g-function

$$\log g(M_0 R) = \log Z_{\mathcal{M}} - \log Z_{\mathcal{M}}^{(CFT)}$$

- Computed in flat space or any conformally equivalent manifold $\mathscr{M}$ (e.g. $\mathbb{R} \times S^{d-1}$ or S^d)
- At the fixed points $R \to 0, \infty$ reduces to a pure number (in a sense that will be clarified soon)
- Reflection positivity demands $g \geq 0$. Invertible topological lines have $g = 1$.



Is g unambiguous?

We are always allowed to shift by a cosmological constant counterterm:

$$\log g \to \log g + \#\, \underbrace{M_0 \int d\sigma}_{\sim M_0 R}$$

To obtain a scheme-independent observable we define the ***defect entropy***:

$$s(M_0 R) = \left(1 - R\frac{\partial}{\partial R}\right) \log g(M_0 R)$$

- At the fixed points s coincides with the universal part g, denoted g_{UV} and g_{IR}
- In $d = 2$ it coincides with the interface contribution to the thermal entropy

DRG and the partition function

For a constant background dilaton

$$s = s\left(M_0 R e^{\Phi}\right)$$

Spurion relates infinitesimal DRG rescaling with local correlation functions

$$M_0 \frac{\partial}{\partial M_0} s(M_0 R) = R \int_{\mathcal{D}} d\phi \, \langle T_D(\phi) \rangle - R^2 \int_{\mathcal{D}} d\phi_1 \int_{\mathcal{D}} d\phi_2 \langle T_D(\phi_1) T_D(\phi_2) \rangle \, .$$

?

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Recall from before:

$$R \int_{\mathcal{D}} d\phi \langle T_D(\phi) \rangle = R^2 \int_{\mathcal{D}} d\phi_1 \int_{\mathcal{D}} d\phi_2 \langle T_D(\phi_1) T_D(\phi_2) \rangle \cos(\phi_1 - \phi_2)$$

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$$\boxed{M_0 \frac{\partial s}{\partial M_0} = -R^2 \int_{\mathcal{D}} d\phi_1 \int_{\mathcal{D}} d\phi_2 \langle T_D(\phi_1) T_D(\phi_2) \rangle \left[1 - \cos\left(\phi_1 - \phi_2\right)\right]}$$

The gradient formula

$$\boxed{R\frac{\partial s}{\partial R} = -R^2 \int_{\mathcal{D}} d\phi_1 \int_{\mathcal{D}} d\phi_2 \langle T_D(\phi_1) T_D(\phi_2)\rangle \left[1 - \cos\left(\phi_1 - \phi_2\right)\right]}$$

- Manifestly finite and unambiguous due to the double zero of [...]
- RHS manifestly negative by unitarity: s monotonically decreases under DRG!
- Integrating between the fixed points $R = 0 \to R = \infty$ we find

$$\boxed{g_{UV} > g_{IR}}$$

Some comments

Reproduces earlier results in $d = 2$

Affleck Ludwig 1991, Witten 1992, Shatasvili 1993, Kutasov Mariño Moore 2000, Friedan Konechny 2003, Casini Salazar-Landea Torroba 2016…

$g_{UV} > g_{IR}$ recently conjectured for arbitrary dimensions

Kobayashi Nishioka Sato Watanabe 2018

Relation with the entanglement entropy:

Lewkowycz Maldacena 2013

$$s_{EE} = s - 2\pi\Omega_{d-2}h_{\mathcal{D}} \,, \qquad \langle T_{dd}(x)\rangle = \frac{h_{\mathcal{D}}}{r^d}$$

- In $d = 2$ $s_{EE} = s$, while in higher dimensions $h_{\mathscr{D}} \neq 0$ and $s_{EE} \neq s$.

Examples

Example 1: line source in free theory

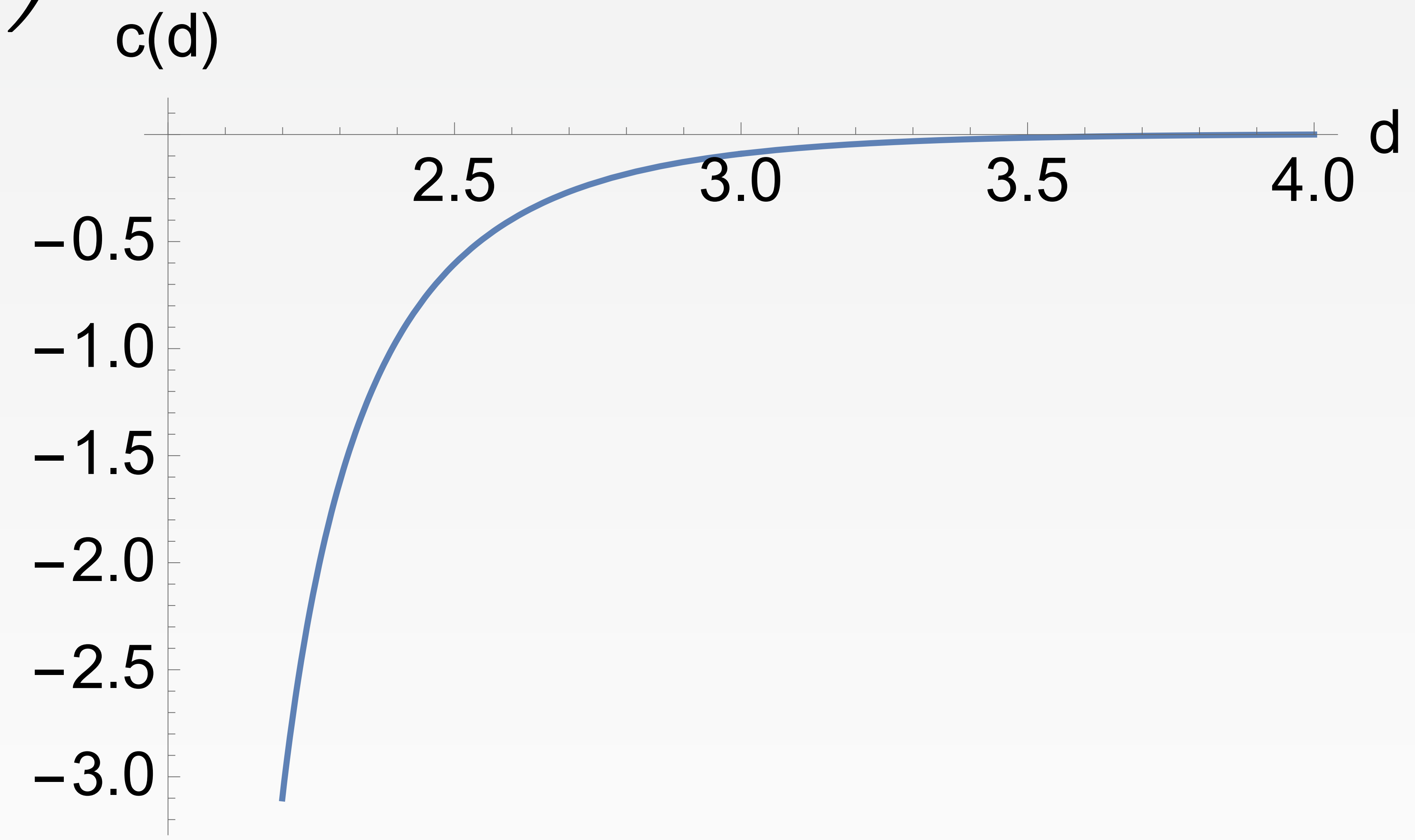
Consider a free massless scalar ϕ in $2 < d \leq 4$ with

$$\mathcal{D} = \exp\left(\lambda M^{\frac{4-d}{2}} \int_{\mathcal{D}} d\sigma \phi\right)$$

Solving for the classical profile sourced by $\mathscr{D}$:

$$s = (1 - R\partial_R) \log g = \pi\lambda^2 c(d)(RM)^{4-d}$$

$$c(3) = -\frac{1}{2\pi^{3/2}}\,, \qquad c(4) = 0$$



- Notice $s \xrightarrow{R\to\infty} -\infty$ hence $g_{IR} = 0$: presumably related to the moduli space
- Result in agreement with the gradient formula ✓

Example 2: localized magnetic field

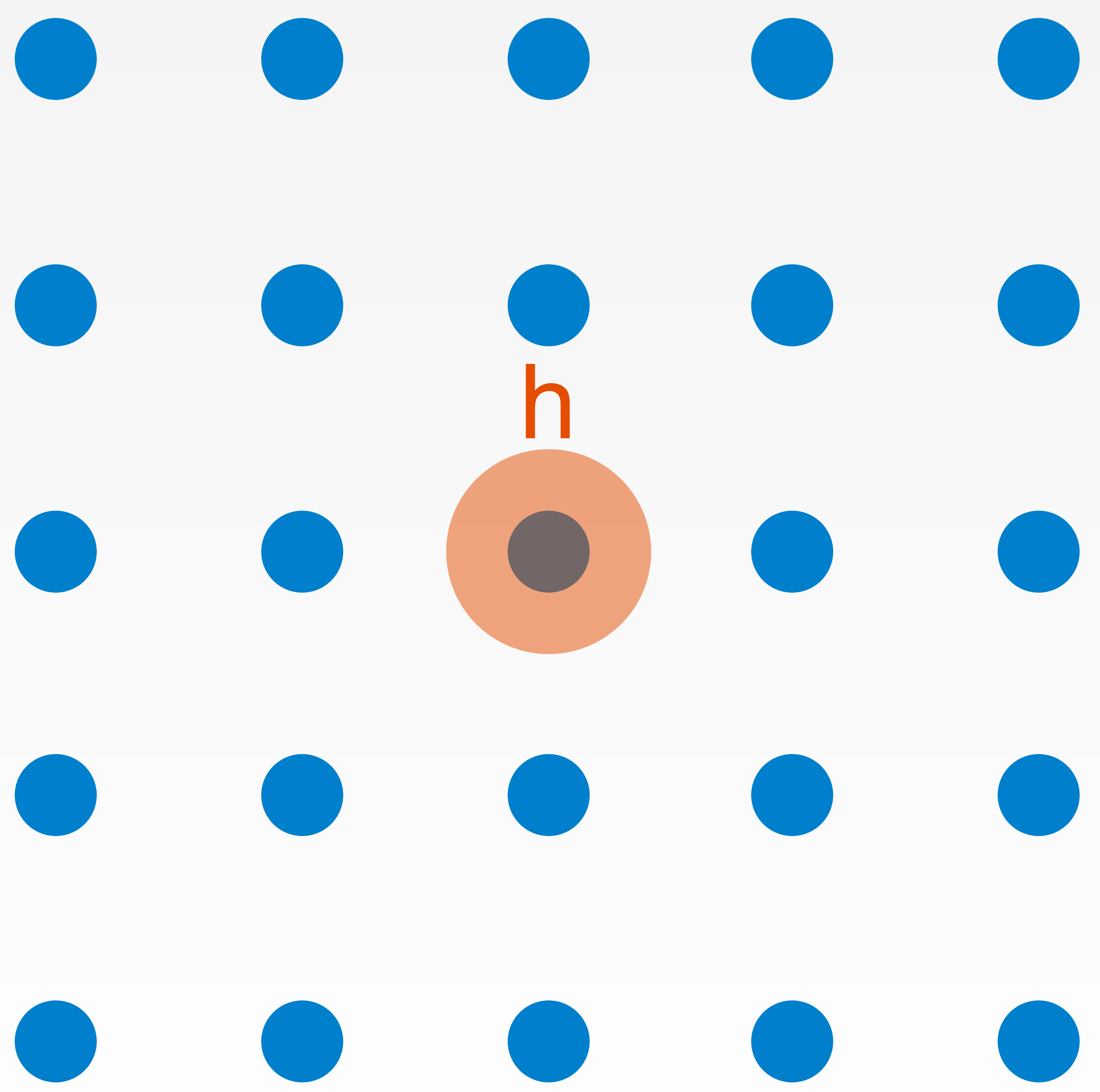
Consider the bulk quantum critical O(N) model in $d = 4 - \varepsilon$:

$$S = \int d^d x \left[\frac{1}{2} (\partial \phi_a)^2 + \frac{\lambda_*}{4!} (\phi_a^2)^2 \right] , \qquad \frac{\lambda_*}{(4\pi)^2} \simeq \frac{3\varepsilon}{N+8}$$

A localized magnetic field perturbation can be interpreted as a symmetry breaking defect:

$$\mathcal{D} = \exp \left(h \int_{\mathcal{D}} d\tau \phi_1 \right)$$

- The trivial fix. pt. $h = 0$ is unstable towards an interacting DCFT in the IR



The model can be studied perturbatively in $\lambda \sim \varepsilon$

- Beta function of h:

$$\beta_h = -\frac{\varepsilon}{2}h + \frac{\lambda_* h^3}{(4\pi)^2 6} + \mathcal{O}\left(\lambda_*^2\right)$$

$$\beta_h = 0 \qquad \Longrightarrow \qquad h_* \simeq 3\varepsilon\frac{(4\pi)^2}{\lambda_*} = N + 8$$

$$h = 0 \xrightarrow{\textit{DRG}} h = h_*$$

Cuomo Komargodski Mezei Raviv-Moshe in progress

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- Defect partition function ($h = h(MR)$) :

$$s \;\simeq\; -\frac{\varepsilon}{8} h^2 + \frac{\lambda_* h^4}{768\pi^2} \overset{\text{fix.pt.}}{=} -\frac{8+N}{16}\varepsilon < \; 0$$

$$h = 0 \xrightarrow{\textit{DRG}} h = h_*$$

- Result in agreement with the gradient formula

Cuomo Komargodski Mezei Raviv-Moshe in progress

Example 3: Wilson-loop flow in $\mathcal{N} = 4$

Consider the following family of fundamental Wilson loops (WLs) in $\mathcal{N} = 4$ SYM:

$$W^{(\zeta)} = \mathrm{Tr}\,\mathcal{P} \exp \oint_C d\tau \left[i A_\mu(x) \dot{x}^\mu + \zeta \Phi_m(x) \theta^m |\dot{x}| \right] , \qquad \theta_m^2 = 1$$

$$\zeta = 0 \xrightarrow{\mathit{DRG}} \zeta = 1$$

The theory admits 2 fixed points:

- $\zeta = 0$: standard Wilson loop
- $\zeta = 1$: 1/2 BPS Maldacena Wilson loop

At small 't Hooft coupling one can study the DRG flow perturbatively.
We find perfect agreement with the gradient formula. ✓

Polchinski Sully 2011, Beccaria Giombi Tseytlin 2017

Conclusions and outlook

Outlook

- The $d > 2$ g-function differs from the entanglement entropy. How to connect with the information-theoretic approach?

  Casini Salazar-Landea Torroba 2016 & 2018

- Is there a lower bound on $\log g$? Under which assumptions?

  Friedan Konechny Schmidt-Colinet 2012

- How to understand flows triggered by VEV of defect operators? These cannot be realized on the circle, challenging the g-theorem.

  Kumar Silvani 2017 & 2018

- More applications to impurities in quantum critical systems?

  Sachdev Buragohain Vojta 1999, Liu Shapourian Vishwanath Metlitski 2021,…

THANK YOU!

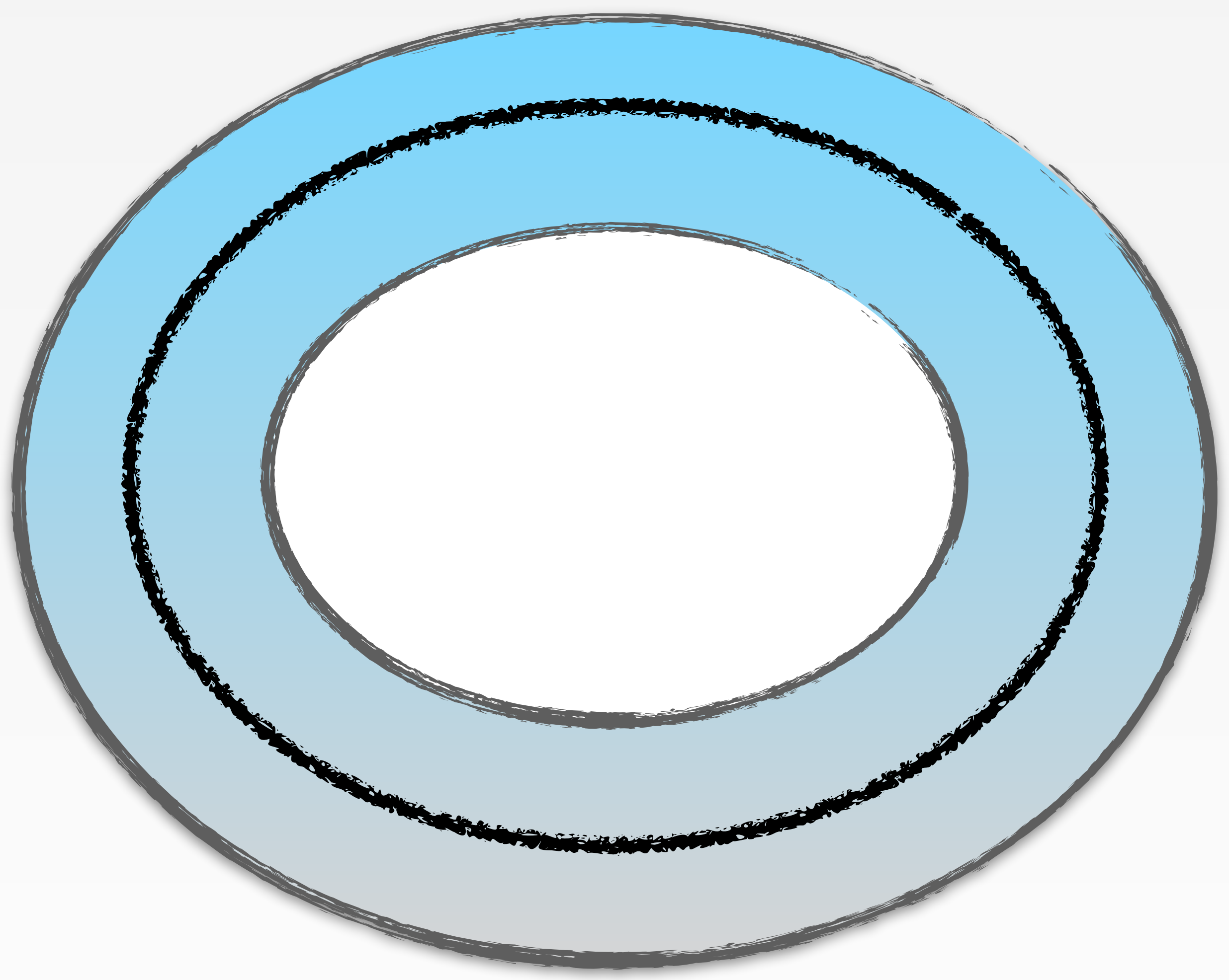
Backup slides

Bulk conformal invariance

Conformal invariance of the vacuum requires for any defect

$$\langle Q_\xi(\mathcal{D})\rangle = 0$$

Proof:



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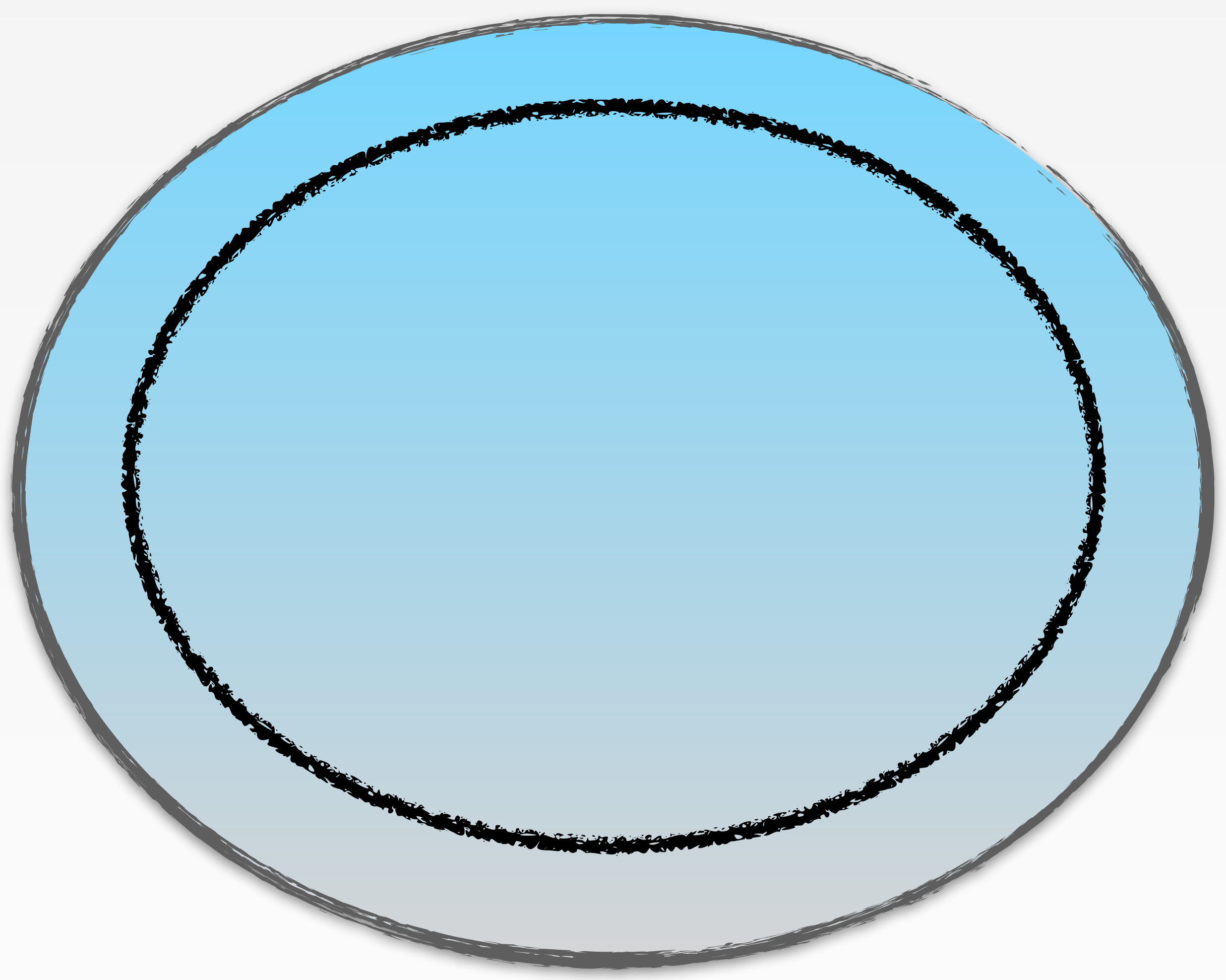

$$= \langle 0|Q_\xi|\mathcal{D}\rangle$$

For a non-conformal defect this identity constraints the DRG flow.

Let us prove the last identity.

- The linear response to the dilaton is given by the defect stress tensor

$$\log Z|_{\Phi+\delta\Phi} = \log Z|_{\Phi} + \int_{\mathcal{D}} d\sigma \delta\Phi(\sigma)\langle T_D(\sigma)\rangle_{\Phi} + \dots$$

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- The source $\Phi(\sigma)$ determines *non-conservation rate* of the charge associated with translations along the defect:

$$\nabla_\mu T_b^{\mu\nu} = -\delta_{\mathcal{D}}^{d-1} \dot{X}^\nu \left(\dot{T}_D - \dot{\Phi} T_D \right) - \delta_{\mathcal{D}}^{d-1} n_i^\nu D^i$$

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- Gauss's law let us evaluate the charge action:

$$\langle Q_\xi(D) \rangle = \int d^{d-1} \Sigma^\mu \langle T_{\mu\nu}^b \rangle \xi^\nu = \int_D d\sigma \left(\dot{\xi}_{\mathcal{D}} + \xi_{\mathcal{D}} \dot{\Phi} \right) \langle T_D \rangle$$

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$$\langle Q_\xi(D)\rangle = \int d^{d-1}\Sigma^\mu \langle T^b_{\mu\nu}\rangle \xi^\nu = \int_D d\sigma \left(\dot{\xi}_{\mathcal{D}} + \xi_{\mathcal{D}}\dot{\Phi}\right) \langle T_D\rangle$$

An additional ambiguity exists in $d = 2$:

$$\log g \sim \log g + \alpha \int d\sigma K$$

Extrinsic curvature

- In $d > 2$ $K = \sqrt{K^i K_i}$ is disallowed since not analytic around flat space
- $K = 0$ for a maximal circle in $\mathbb{R} \times S^{d-1}$ or S^d
- CPT implies $\mathrm{Re}[\alpha] = 0$ $\longrightarrow$ $\mathrm{Re}[s]$ is *unambiguous*

Chang Lin Shao Wang Yin 2018

Example 3: Wilson-loop flow in $\mathcal{N} = 4$

Consider the following family of fundamental Wilson loops (WLs) in $\mathcal{N} = 4$ SYM:

$$W^{(\zeta)} = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \oint_C d\tau \left[i A_\mu(x) \dot{x}^\mu + \zeta \Phi_m(x) \theta^m |\dot{x}| \right] , \quad \theta_m^2 = 1$$

The theory admits 2 fixed points:

- $\zeta = 0$: standard Wilson loop
- $\zeta = 1$: 1/2 BPS Maldacena Wilson loop

$$\zeta = 0 \xrightarrow{\mathit{DRG}} \zeta = 1$$

At small 't Hooft coupling one can study the DRG flow perturbatively

Polchinski Sully 2011, Beccaria Giombi Tseytlin 2017

For $\lambda = g^2 N \ll 1$

Polchinski Sully 2011

$$\beta_\zeta = M_0 \frac{\partial \zeta}{\partial M_0} = -\frac{\lambda}{8\pi^2} \zeta \left(1 - \zeta^2\right) + \mathcal{O}\left(\lambda^2\right)$$

For $\lambda = g^2 N \ll 1$

Polchinski Sully 2011

$$\beta_\zeta = M_0 \frac{\partial \zeta}{\partial M_0} = -\frac{\lambda}{8\pi^2} \zeta \left(1 - \zeta^2\right) + \mathcal{O}\left(\lambda^2\right)$$

The expectation value of the WL depends on $M_0 R$ through $\zeta = \zeta(M_0 R)$:

$$\langle W^{(\zeta)} \rangle \equiv W\left(\lambda; \zeta(M_0 R), M_0 R\right) , \qquad M_0 \frac{\partial}{\partial M_0} W + \beta_\zeta \frac{\partial}{\partial \zeta} W = 0$$

Explicitly one finds

Beccaria Giombi Tseytlin 2017

$$\langle W^{(\zeta)} \rangle = 1 + \frac{1}{8}\lambda + \left[\frac{1}{192} + \frac{1}{128\pi^2}\left(1 - \zeta^2\right)^2\right] \lambda^2 + \mathcal{O}(\lambda^3)$$

- Results perturbative in λ but exact throughout the flow

$$\langle W^{(\zeta)} \rangle = 1 + \frac{1}{8}\lambda + \left[\frac{1}{192} + \frac{1}{128\pi^2}\left(1 - \zeta^2\right)^2\right]\lambda^2 + \mathcal{O}(\lambda^3)$$

- In agreement with $g_{UV} > g_{IR}$:

$$\log\langle W^{(0)} \rangle - \log\langle W^{(1)} \rangle = \frac{\lambda^2}{128\pi^2} > 0$$

$$\langle W^{(\zeta)} \rangle = 1 + \frac{1}{8}\lambda + \left[\frac{1}{192} + \frac{1}{128\pi^2}\left(1-\zeta^2\right)^2\right]\lambda^2 + \mathcal{O}(\lambda^3)$$

- In agreement with $g_{UV} > g_{IR}$:

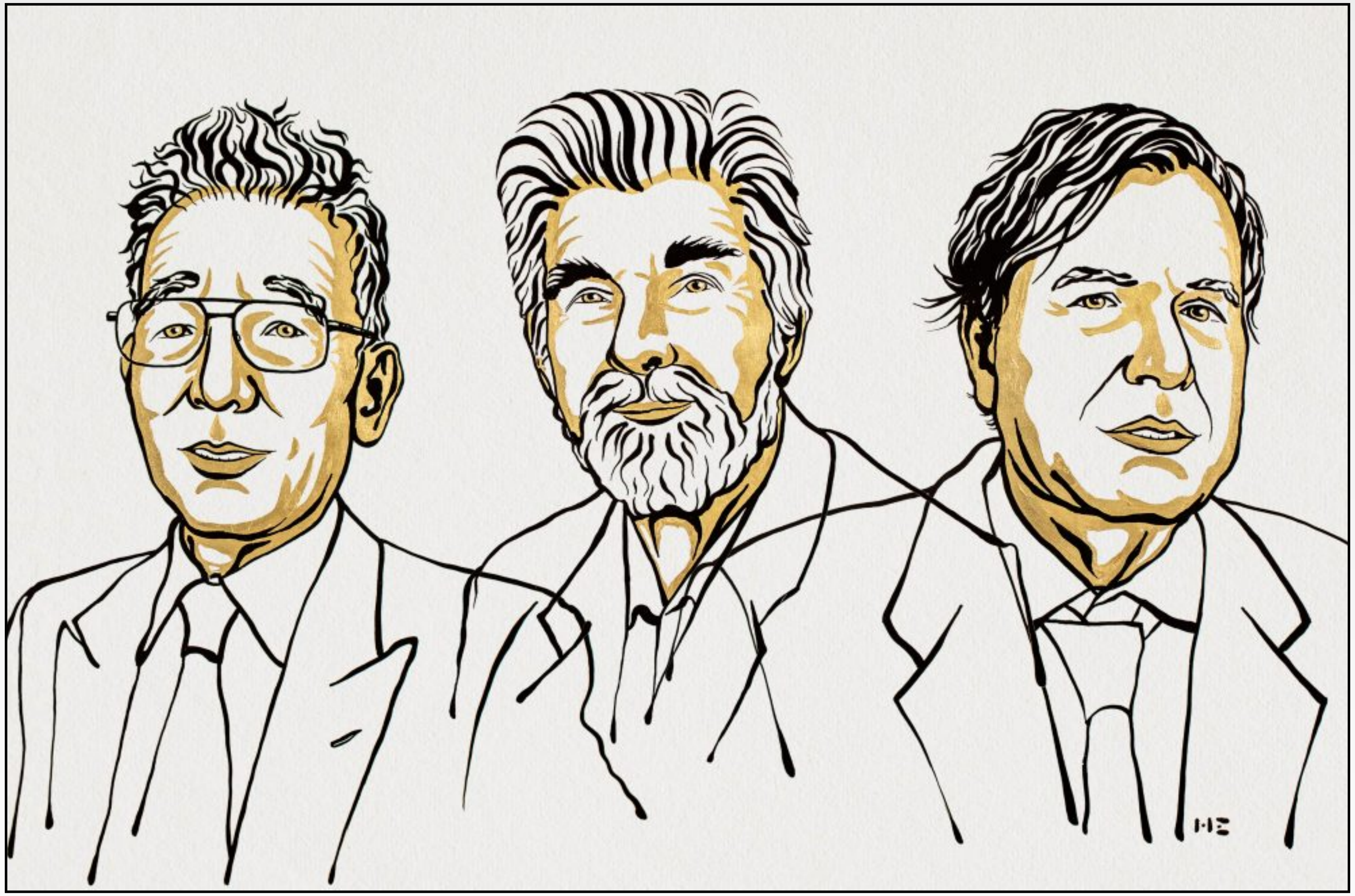
$$\log\langle W^{(0)}\rangle - \log\langle W^{(1)}\rangle = \frac{\lambda^2}{128\pi^2} > 0$$

The DRG gradient of the defect entropy is:

$$M_0 \frac{\partial s}{\partial M_0} = -\frac{\lambda^3}{256\pi^4}\zeta^2\left(1-\zeta^2\right)^2 + \mathcal{O}\left(\lambda^4\right).$$

- Agreement with the gradient formula follows from the defect stress tensor:

$$T_D = \beta_\zeta \theta_m \Phi^m \quad \Longrightarrow \quad \langle T_D(\phi) T_D(0)\rangle = \frac{\lambda}{8\pi^2}\frac{\beta_\zeta^2}{\left(2\sin\frac{\phi}{2}\right)^2}\left[1 + \mathcal{O}\left(\lambda\right)\right]$$



CONGRATULATIONS!