Entanglement in Matrix Quantum Mechanics

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Strings, fields & holograms (Ascona, virtually)
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Motivation

Some major themes in 20th century theoretical physics:

- Entropy counts microscopic degrees of freedom! [Boltzmann]
- Black holes have entropy! [Bekenstein and Hawking]
- In certain cases can be matched microscopically! [Strominger and Vafa: SUSY + CFT]

Motivation

Quantum entanglement promises to be an organizing principle for 21st century physics:

- Local interactions ↔ area law entanglement [Bombelli et al., Srednicki, …]

  Gauge interactions ↔'topological' entanglement [Kitaev-Preskill, Levin-Wen, …]

- Spacetime requires a lot of entanglement [Ryu-Takayangi, Lewkowyck-Maldacena, …]

- What is the microscopic (bulk) origin of this entanglement? How is spacetime actually made? <u>Needed: Strominger-Vafa for the 21st century</u>.

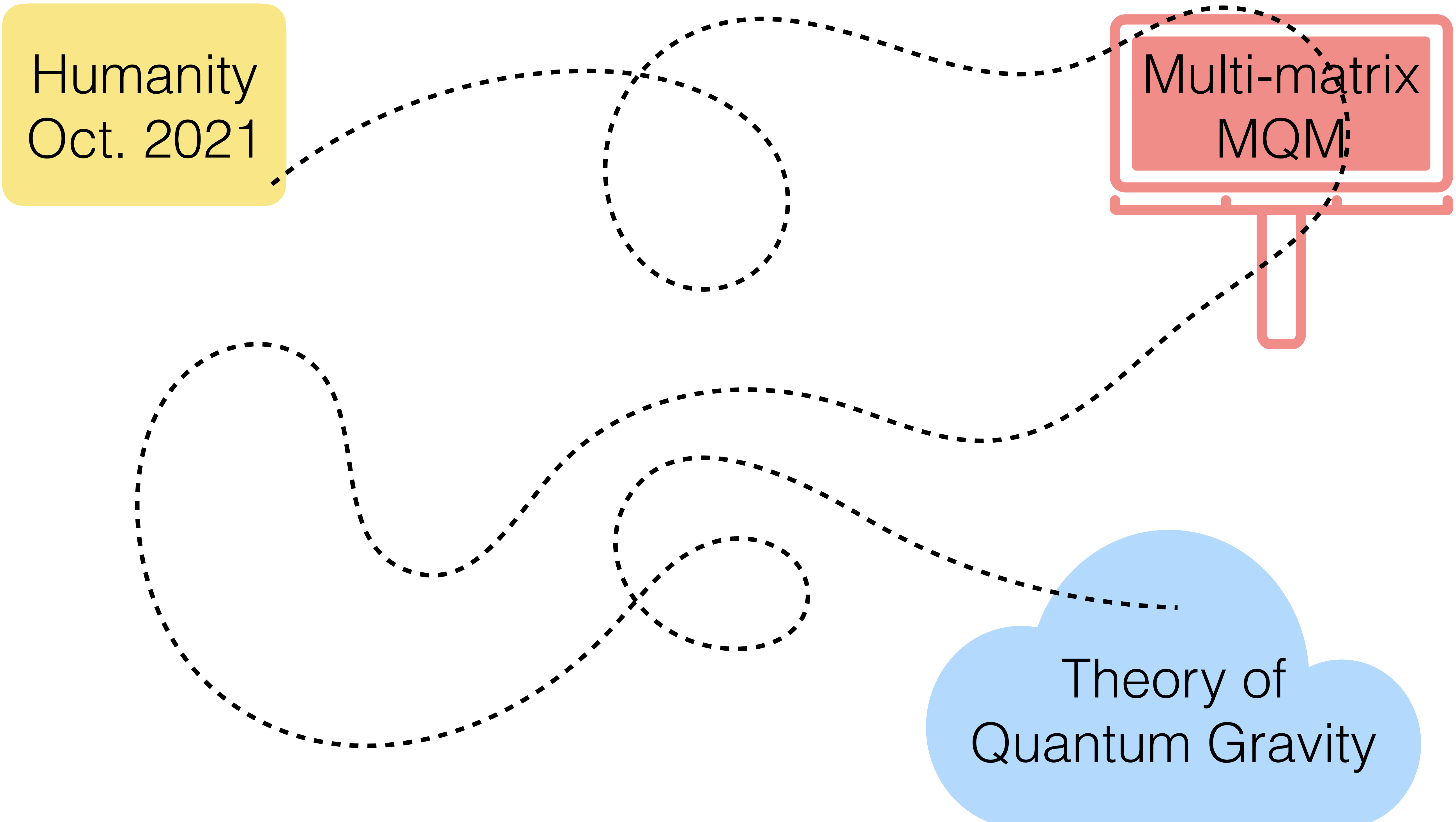
Motivation

- Tensor networks give a ‘skeleton’ of spacetime that is built using boundary locality, which is already manifest [Swingle, …]

- The ‘flesh’ of spacetime, however, is due to microscopic models that support a fully emergent locality. Want to understand entanglement.

- Best understood framework: large N matrices.

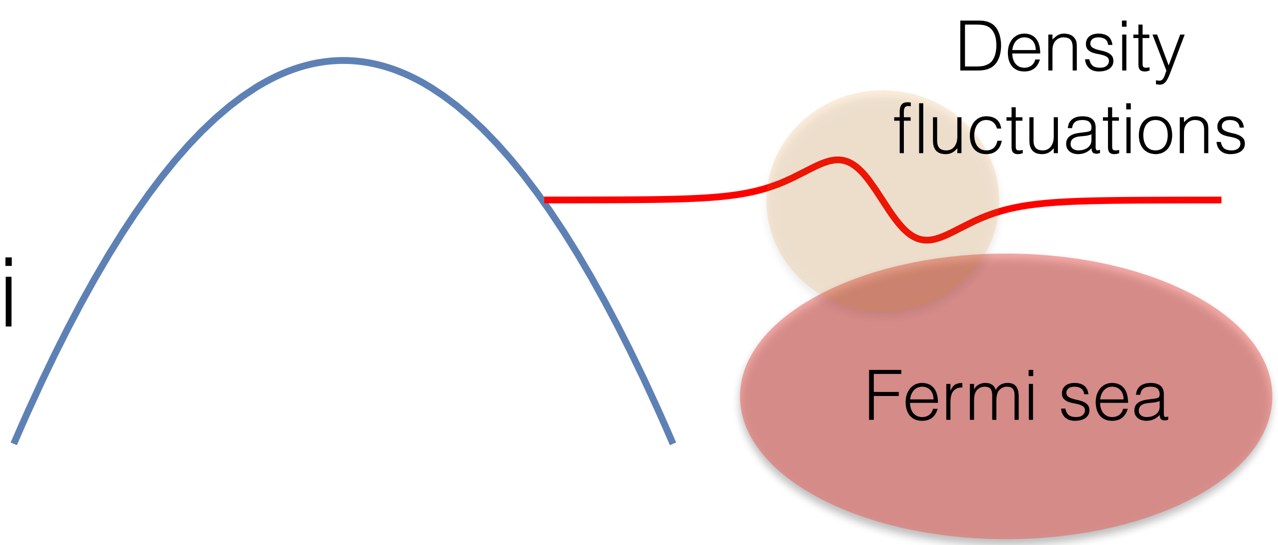
Motivation


Humanity
Oct. 2021
Multi-matrix
MQM
Theory of
Quantum Gravity

Baby model: single matrix

- Singlet sector of $\mathcal{L} \sim \mathrm{tr}\left[\dot{M}^2 - V(M)\right]$ described by eigenvalues {λ} of M.

- Eigenvalues are non-interacting fermions. Fermi sea builds 1d space.



- Entanglement of interval [λ_1,λ_2] using conventional many-body methods. Matches emergent 1+1 'tachyon' field [Das 95, Hartnoll-Mazenc 15]:

$$S_{[\lambda_1,\lambda_2]} = \frac{1}{3} \log \frac{\tau(\lambda_2) - \tau(\lambda_1)}{\sqrt{g_s(\lambda_1) g_s(\lambda_2)}/\mu}$$

Beyond one matrix

- Eigenvalues are not enough. ‘Off-diagonal’ modes stretching between coincident branes essential for ‘grown up’ holography.
- Noted by [Das-Kaushal-Mandal-Trivedi 20] that a class of proto-geometric partitions are obtained by diagonalizing one of the matrices (e.g. X_1).
- Eigenvalues of X_1 dealt with as in the single matrix case. Induces a block decomposition of the remaining matrices. Various proposals made for dealing with the off-diagonal blocks.

Plan

- Consider a solvable matrix quantum mechanics with two matrices.
- Compute the entanglement of a geometric partition. New treatment of off-diagonal modes inspired by entanglement in gauge theories.
- Obtain emergent 2d 'area law' and topological-like subleading correction.

Work to appear shortly with Alex Frenkel, also many discussions with Xizhi Han and Onkar Parrikar.

Quantum Hall Matrix Model

- Quantum Hall phases: incompressible droplet supporting emergent Chern-Simons dynamics.

- Minimal microscopic realization: discretize the area-preserving diffeos of the droplet into U(N). [Susskind 01]

- IR-regulated version by [Polychronakos 01]:

$$H = \mathrm{tr}\left(X^2 + Y^2\right) \qquad [X_{ab}, Y_{cd}] = i\delta_{ad}\delta_{bc}$$

(Gauss law) $$-i[X,Y] + \Psi\Psi^\dagger = k$$

Quantum Hall Matrix Model

- Ground state [Hellerman-Van Raamsdonk 01]:

$$|\psi\rangle = \left[\epsilon^{a_1 \dots a_N} \Psi^\dagger_{a_1} (\Psi^\dagger Z^\dagger)_{a_2} \cdots (\Psi^\dagger Z^{\dagger\, N-1})_{a_N}\right]^k |0\rangle$$

Here Z = X + iY.

- State simple in terms of variables {x,U,Ψ} where

$$X = U x U^\dagger \,, \qquad \Psi = U \widetilde{\Psi}$$

$$\psi = (\det U)^k \prod_{a<b} (x_a - x_b)^k e^{-\frac{1}{2}\sum_d x_d^2} \prod_c \widetilde{\Psi}_c^k e^{-\frac{1}{2}\sum_d |\widetilde{\Psi}_d|^2}$$

[Karabali-Sakita 01]

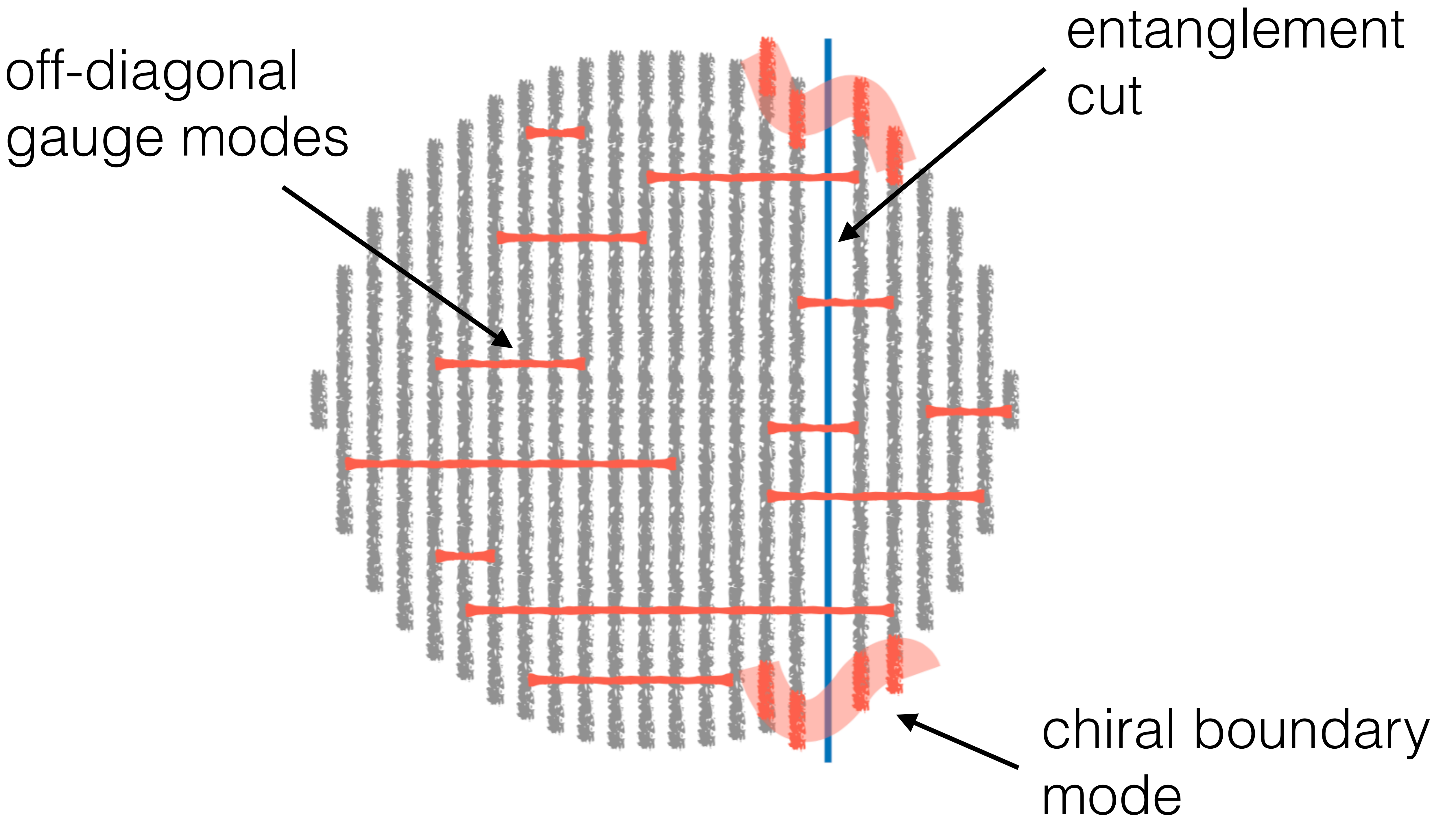
Quantum Hall Matrix Model

- Wavefunction factorizes — allows computation of two contributions to the entanglement due to a vertical (fixed X) partition of the droplet:

  (1) A 'collective field' contribution from fluctuations of the x eigenvalues. Physically: correlations due to chiral boundary mode.

  (2) A 'gauge theoretic' contribution from an associated block partition of the U. Physically: nonlocal correlations due to the Gauss law.

Quantum Hall Matrix Model



Collective field entropy

- Similar to computations of the entanglement in single-matrix models. But we used a new method.

- In terms of the collective field $n(x) \equiv \sum_a \delta(x - x_a)$

  the wavefunction $\psi[n] = e^{S[n]}$

$$S[n] = \frac{k+1}{2} \int dx_1 dx_2 n(x_1) n(x_2) \log |x_1 - x_2| - \frac{1}{2} \int dx n(x) x^2$$

  is strongly peaked on the Wigner semi-circle:

$$n_o(x) = \frac{2N}{\pi R^2} \sqrt{R^2 - x^2}\,, \qquad R^2 = 2N(k+1)$$

Collective field entropy

- Fluctuations about the semi-circle $n(x) = n_o(x) + \delta n(x)$ are described by the Gaussian wavefunction:

$$\psi[\delta n] = e^{\frac{k+1}{2}\int dx_1 dx_2 \delta n(x_1)\delta n(x_2)\log|x_1 - x_2|}$$

- Using steps from [Jackiw-Strominger 81] one can express this wave function in terms of a chiral boson ϕ:

$$\psi[\delta n] = \int \mathcal{D}\phi e^{-\int d\tau d\theta [i\partial_\tau \phi \partial_\theta \phi + (\partial_\theta \phi)^2] - i\int d\theta \phi(\theta)\delta n(\theta)}$$

Collective field entropy

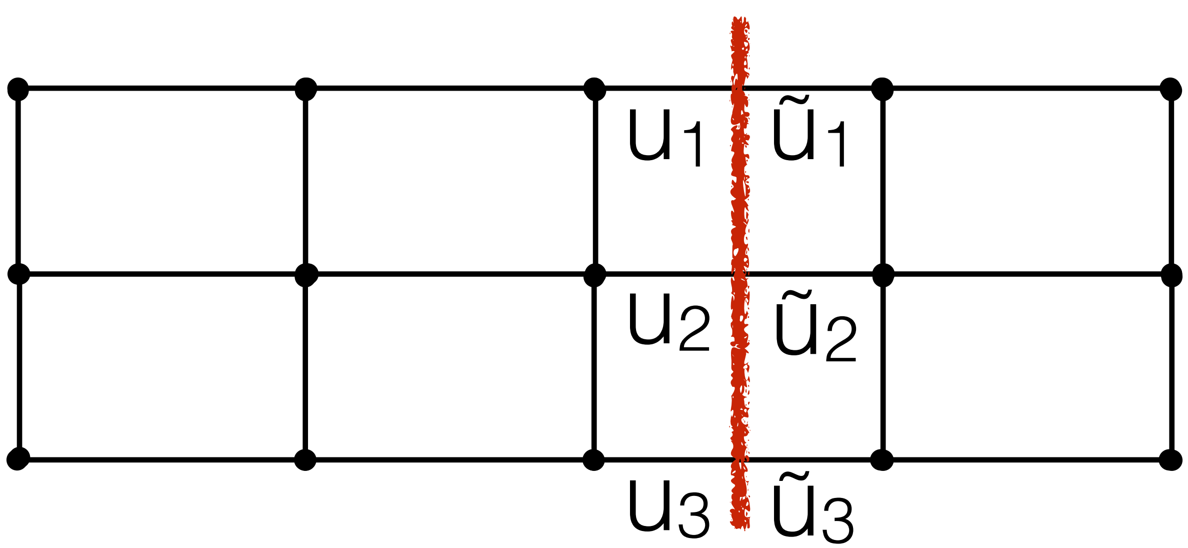
- Can show that the ‘target space’ entanglement of the eigenvalues is equal to the usual entanglement of the chiral boson.
- This is fixed by conformal invariance.
- Finite N cuts off the mode expansion of the boundary chiral field so that:

$$s_{\rm bdy} = \frac{1}{6}\log(NL) + \cdots$$

↑ Length of cut with radius normalized to one.

Lattice gauge theory

- The gauge-invariant variables (Wilson loops) of a gauge theory are not local. Defining a geometric partition therefore requires some work.
- [Donnelly 12]: links crossing the boundary are assigned to both regions. In simple situations the two copies of the holonomy on the link are required by gauge invariance to be maximally entangled.



$$\otimes_i \left(\sum_{u_i} |u_i\rangle |u_i\rangle \right) \quad \Rightarrow \quad \Delta s \sim L \, \log(\dim R)$$

Partitioning the matrices

- In the full theory, the U degrees of freedom are 'pure gauge' and do (almost) nothing.
- We will see however, that when the system is partitioned some of the U's acquire dynamics.
- We partition U in the spirit of [Donnelly-Freidel 16]. I.e. write the kinetic term (which determines symplectic structure) in the Lagrangian as a sum:

$$L_{\text{kin}} = i\,\text{tr}\left(\Theta Z^\dagger \frac{d}{dt} Z + (1-\Theta) Z^\dagger \frac{d}{dt} Z\right) = L_{L\,\text{kin}} + L_{R\,\text{kin}}$$

Partitioning the matrices

- Here Θ projects to the space of lowest M eigenvalues of X. One finds:

$$L_{L\,\mathrm{kin}} = y_L \frac{d}{dt} x_L + \frac{i}{2}\,\mathrm{tr}\left(Y_{RL}^\dagger \frac{d}{dt} Y_{RL}\right)$$

$$L_{R\,\mathrm{kin}} = y_R \frac{d}{dt} x_R + \frac{i}{2}\,\mathrm{tr}\left(Y_{LR}^\dagger \frac{d}{dt} Y_{LR}\right)$$

These are the off-diagonal modes that live in both regions

- The Gauss law fixes

$$Y_{LR}^\dagger = Y_{RL} = U_R Y_{RL}^{\mathrm{cl}} U_L^\dagger \qquad Y_{ab}^{\mathrm{cl}} = y_a \delta_{ab} - i \frac{\widetilde{\Psi}_a^\dagger \widetilde{\Psi}_b}{x_a - x_b}\,.$$

(It is important only to consider $U = U_L U_R$ that respect the block partition)

Partitioning the matrices

To make a long-ish story short (details on arXiv shortly!):

- On each side of the cut U_L and U_R become families of harmonic oscillators, constrained to fixed energy.
- The energy is set by the singular values of $Y_{RL}^{\rm cl}$.
- Gauge invariance forces these oscillators to be maximally entangled across the cut.
- The oscillators are ‘identical’. This is inherited from the fact that U must not include permutations that re-order the eigenvalues.

Gauge theoretic entropy

- With the above in place, the gauge-theoretic entanglement entropy can be computed using the Hardy-Ramanujan formula to count the dimension of the entangled oscillator space. We find:

$$s = \frac{(Nk)^{1/2}\log(NL)}{\sqrt{6}} L + \cdots$$

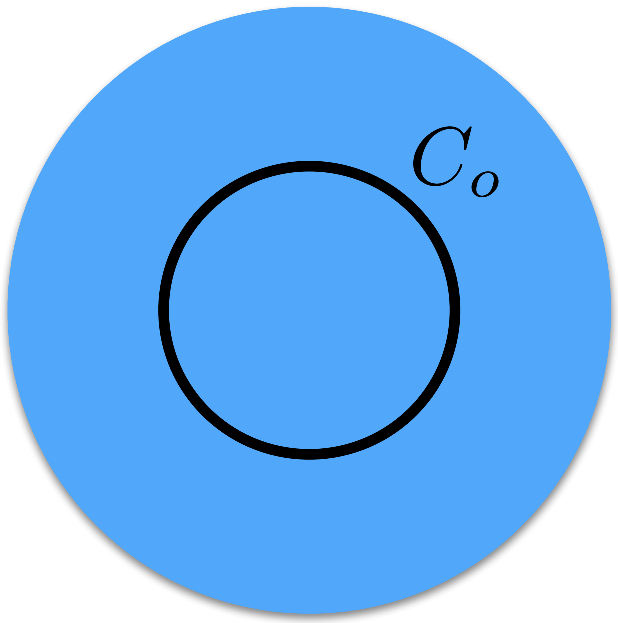
- Area law, regulated by finite N — emergent geometry!

- Logarithmic violation due to L-dependent cutoff on the values the emergent U(1) connection can take.

Radial partition

- We also looked at a radial partition, using a similar framework.
Find:

$$s = \frac{(Nk)^{1/2}}{\sqrt{6}} C_o - 2\log[(Nk)^{1/2} C_o] + \cdots$$



- No logarithmic violation of area law here, because there is more symmetry in the cutoff.

- Trust subleading term in this case. Reminiscent of topological entanglement terms.

Conclusions

- What partition of matrix degrees of freedom captures the partition of an emergent geometry?
- The matrix Hamiltonian does not have spatial locality. But the wavefunction should contain an emergent locality.
- We have defined a partition in a very simple two-matrix model and computed the corresponding entropy (subtleties: role of permutations and ‘partial gauge fixing’).

Conclusions

- We found two contributions that match the expected emergent locality:

  1. a logarithmic entanglement from eigenvalues
     → chiral boundary mode
  2. an area-law gauge-theoretic entanglement
     → bulk Chern-Simons field

- Now have the understanding to move on to a more complicated model with compressible bulk dynamics.