Recent Developments in N=4 Yang-Mills Amplitudes

Anastasia Volovich

Brown University

Mago, Ren, Schreiber, Spradlin, Yelleshpur Srikant

2007.00646, 2012.15812, 2106.01405, 2106.01406





Planar N=4 Yang-Mills Amplitudes

- Planar N=4 Yang-Mills scattering amplitudes have been computed to very high loop order.

- They have many remarkable properties, that have sparked interest from mathematicians working on combinatorics, algebraic geometry and number theory.

- At the same time, several methods that have been developed for N=4 Yang-Mills are directly applicable to, and have greatly aided, QCD computations.

Outline

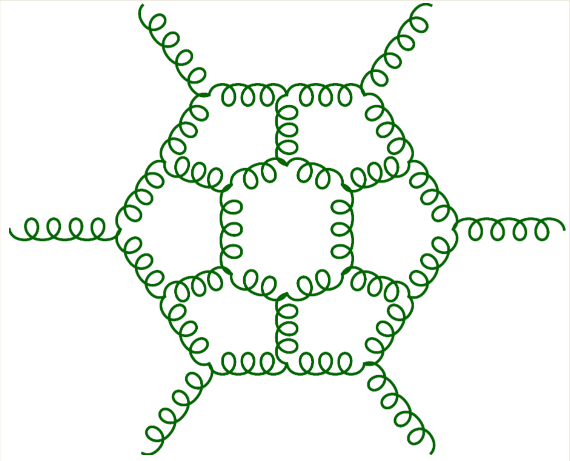
- Status, methods and tools for amplitude computations
- Survey of singularities for known amplitudes
- 6 and 7-point amplitudes: singularities from cluster algebras
- 8 and 9-point amplitudes: new features beyond cluster algebras
- Recent work: singularities from plabic graphs and from tensor diagrams

Status: n-point N=4 Yang-Mills amplitudes

- n<6 all loops — Bern, Dixon, Smirnov '05
- n=6 through 7-loops
- n=7 through 4-loops — Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou, review: 2005.06735
- All n MHV through 2-loops — Caron-Huot '11
- n=8 MHV through 3-loop — Li, Zhang'21
- n=8, 9 NMHV through 2-loops — He, Li, Zhang '19'20

Method: amplitudes bootstrap

- Write down the answer as linear combo of functions (based on the singularity structure of amplitudes)
- Determine the coefficients by solving a system of linear constraints.



[Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, Papathanasiou, review: 2005.06735]

Remaining number of parameters after each constraint for 6-point (MHV, NMHV)

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathscr{H}_6$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. N^{3}LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Tools: N=4 Yang-Mills amplitudes

- MHV and NHMV L-loop amplitudes can be expressed in terms of multiple polylogarithms of weight m=2L [beyond NMHV: elliptic].
- The arguments of these polylogarithms encode the location of singularities [symbol alphabet].
- We express kinematics in terms of momentum twistors.

$$\mathbf{Z_i^A = (Z_i^1, Z_i^2, Z_i^3, Z_i^4) \in P^3}$$

$$\langle ijkl \rangle \equiv \langle Z_i Z_j Z_k Z_l \rangle = \det(Z_i Z_j Z_k Z_l)$$

Singularities
[Symbol Alphabet]
for
(so far) known amplitudes

6-point amplitude

Singularities [symbol alphabet] are described by 15 letters which are Gr(4,6) Plucker coordinates

<a a+1 b c>

$$R_6^{(2)} = \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\,\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right) + \cdots$$

Del Duca, Duhr, Smirnov;
Goncharov Spradlin Vergu AV

7-point amplitude

Singularities [symbol alphabet] are described by 49 letters which are

35 Gr(4,7) Plucker coordinates <a a+1 b c>

plus 7 cyclic images of

<1(23)(45)(67)> and <1(27)(34)(56)>

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

$$R_7^{(2)} = \frac{1}{4}\mathrm{Li}_{2,2}\left(\frac{\langle 1267\rangle\langle 2345\rangle}{\langle 1237\rangle\langle 2456\rangle}, -\frac{\langle 2456\rangle\langle 1(23)(45)(67)\rangle}{\langle 1267\rangle\langle 1456\rangle\langle 2345\rangle}\right) - \frac{1}{2}\,\mathrm{Li}_{2,2}\left(\frac{\langle 1267\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1567\rangle}, \frac{\langle 1(27)(34)(56)\rangle}{\langle 1267\rangle\langle 1345\rangle}\right) + \cdot$$

Caron-Huot;
Golden Goncharov Spradlin Vergu AV

8-point amplitude

He, Li, Zhang '19: 2-loop NMHV

180 RATIONAL LETTERS

- 68 Plücker coordinates of the form $\langle a\ a{+}1\ b\ c\rangle$,
- 8 cyclic images of $\langle 12\bar{4}\cap\bar{7}\rangle$,
- 40 cyclic images of $\langle 1(23)(45)(78)\rangle$, $\langle 1(23)(56)(78)\rangle$, $\langle 1(28)(34)(56)\rangle$, $\langle 1(28)(34)(67)\rangle$ $\langle 1(28)(45)(67)\rangle$,
- 48 dihedral images of $\langle 1(23)(45)(67)\rangle$, $\langle 1(23)(45)(68)\rangle$, $\langle 1(28)(34)(57)\rangle$,
- 8 cyclic images of $\langle\bar{2}\cap(245)\cap\bar{8}\cap(856)\rangle$,
- and 8 distinct dihedral images of $\langle\bar{2}\cap(245)\cap\bar{6}\cap(681)\rangle$.

$$\bar{a}\equiv(a{-}1\ a\ a{+}1)$$
$$\langle ab(cde)\cap(fgh)\rangle=\langle acde\rangle\langle bfgh\rangle-\langle bcde\rangle\langle afgh\rangle$$
$$\langle\bar{x}\cap(abc)\cap\bar{y}\cap(def)\rangle\equiv\langle a,(bc)\cap\bar{x},d,(ef)\cap\bar{y}\rangle$$
$$\langle a,b,c,(de)\cap(fgh)\rangle\equiv\langle abcd\rangle\langle efgh\rangle-\langle abce\rangle\langle dfgh\rangle$$

2 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357}=(\langle 1256\rangle\langle 3478\rangle-\langle 1278\rangle\langle 3456\rangle-\langle 1234\rangle\langle 5678\rangle)^2-4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$

and 1 cyclic

8-point amplitude

180 RATIONAL LETTERS

He, Li, Zhang '19: 2-loop NMHV

- 68 Plücker coordinates of the form $\langle a\, a{+}1\, b\, c\rangle$,
- 8 cyclic images of $\langle 12\bar{4} \cap \bar{7}\rangle$,
- 40 cyclic images of $\langle 1(23)(45)(78)\rangle$, $\langle 1(23)(56)(78)\rangle$, $\langle 1(28)(34)(56)\rangle$, $\langle 1(28)(34)(67)\rangle$ $\langle 1(28)(45)(67)\rangle$,
- 48 dihedral images of $\langle 1(23)(45)(67)\rangle$, $\langle 1(23)(45)(68)\rangle$, $\langle 1(28)(34)(57)\rangle$,
- 8 cyclic images of $\langle \bar{2} \cap (245) \cap \bar{8} \cap (856)\rangle$,
- and 8 distinct dihedral images of $\langle \bar{2} \cap (245) \cap \bar{6} \cap (681)\rangle$.

Additional 24 letters 3-loop MHV Li, Zhang'21

$\langle 1(23)(46)(78)\rangle$, $\langle \bar{2} \cap \bar{4} \cap (568) \cap \bar{8}\rangle$ and $\langle \bar{2} \cap \bar{4} \cap \bar{6} \cap (681)\rangle$

2 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357} = (\langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle)^2 - 4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$

and 1 cyclic

9-point amplitude

He, Li, Zhang '20: 2-loop NMHV

59 x 9 RATIONAL LETTERS

- 13 cyclic classes of $\langle 12kl\rangle$ for $3 \leq k < l \leq 8$ but $(k,l) \neq (6,7),(7,8)$;
- 7 cyclic classes of $\langle 12(ijk)\cap(lmn)\rangle$ for $3 \leq i < j < k < l < m < n \leq 9$;
- 8 cyclic classes of $\langle \bar{2}\cap(245)\cap\bar{6}\cap(691)\rangle$, $\langle \bar{2}\cap(346)\cap\bar{6}\cap(892)\rangle$, $\langle \bar{2}\cap(346)\cap\bar{2}\cap(782)\rangle$, $\langle \bar{2}\cap(245)\cap\bar{7}\cap(791)\rangle$, $\langle \bar{2}\cap(245)\cap(568)\cap\bar{8}\rangle$, $\langle \bar{2}\cap(245)\cap(569)\cap\bar{9}\rangle$, $\langle \bar{2}\cap(245)\cap(679)\cap\bar{9}\rangle$, $\langle \bar{2}\cap(256)\cap(679)\cap\bar{9}\rangle$;
- 10 cyclic classes of $\langle 1(i\,i{+}1)(j\,j{+}1)(k\,k{+}1)\rangle$ for $2 \leq i,\ i+1 < j,\ j+1 < k \leq 8$;
- 6 cyclic classes $\langle 1(2i)(j\,j{+}1)(k9)\rangle$ for $3 \leq i < j,\ j+1 < k \leq 8$, but $(i,k) \neq (3,8),(4,7)$;
- 14 cyclic classes of $\langle 1(29)(ij)(k\,k{+}1)\rangle$ for $3 < i < j \leq 8,\ 3 \leq k \leq i-2$ or $j+1 \leq k \leq 7$;
- 1 cyclic class of $\langle 1,(56)\cap\bar{3},(78)\cap\bar{3},9\rangle$.

11 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357} = (\langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle)^2 - 4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$ and 8 cyclic

Is there
a mathematical
description of
these singularities?

Yes: Cluster Algebras.

We observed that singularities are given by subsets of cluster coordinates of Grassmannian cluster algebra

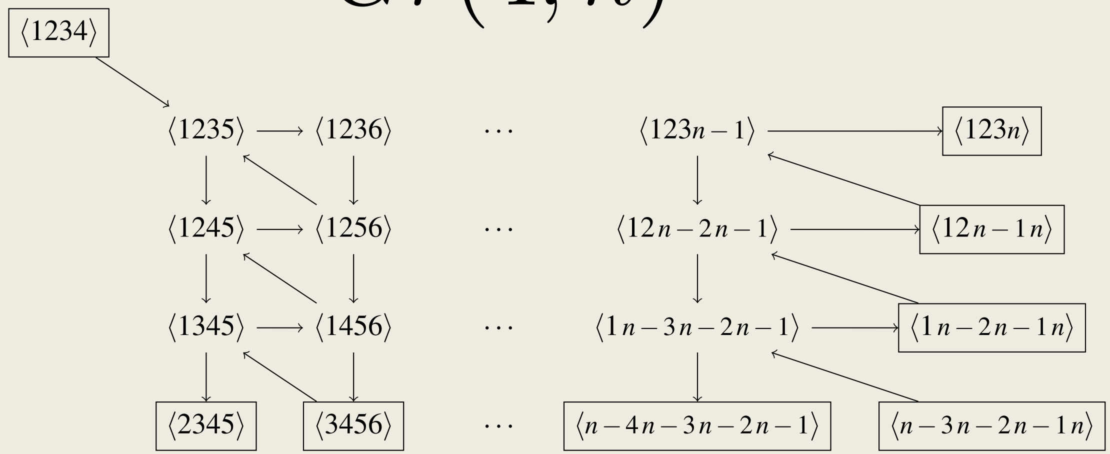
$$Gr(4, n)$$

Golden, Goncharov, Spradlin, Vergu, AV

Grassmannian Cluster Algebra

$$Gr(4,n)$$

Initial Quiver



Mutation Rule

$$a_k \to a_k' = \frac{1}{a_k}\left(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j\right)$$

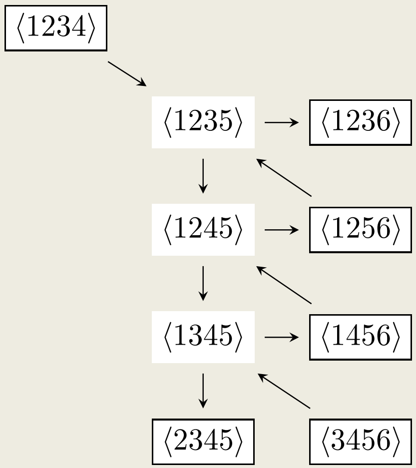
Cluster Coordinates

$$\{a_k\}$$

Fomin, Zelevinsky '02; Scott; Gekhtman, Shapiro, Vainshtein
Cluster Algebra Portal: http://www.math.lsa.umich.edu/~fomin/cluster.html

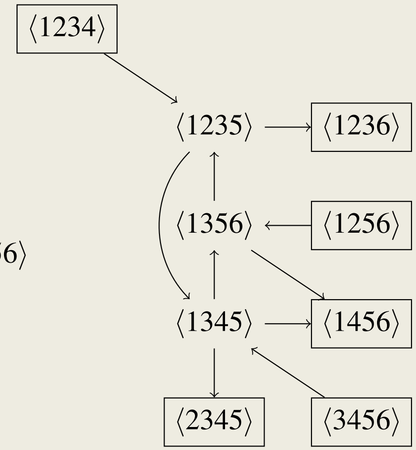
Cluster Coordinates: n=6 and n=7

n=6



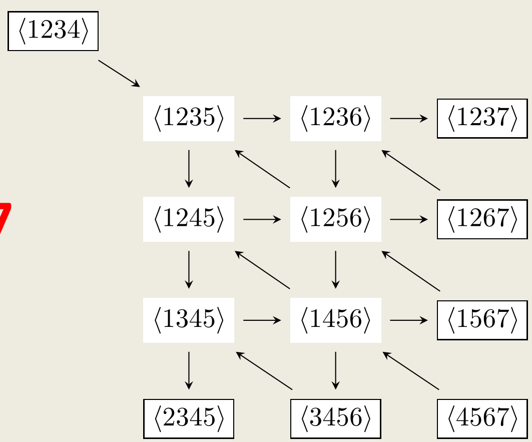
Mutate

⟨1245⟩ → ⟨1356⟩



14 quivers give
15 cluster coordinates
<a a+1 b c>

n=7



833 quivers give
49 cluster coordinates <a a+1 b c>,
7 cyclic images <1(23)(45)(67)>, <1(27)(34)(56)>

Matches singularities for n=6, 7 amplitudes!

Caron-Huot; Golden, Goncharov, Spradlin, Vergu, AV

New Features at n=8

- Gr(4,n) cluster algebra is infinite for n>7

Fomin, Zelevinsky

- Singularities involve square roots

He, Li, Zhang

New Features at n=8

- Gr(4,n) cluster algebra is infinite for n>7
- Singularities involve square roots

What is a mathematical description?

1. Tropical Geometry Drummond, Foster, Gurdogan, Kalousios ’19 Henke, Papathanasiou ’19 ‘21
2. Dual Polytopes Arkani-Hamed, Lam, Spradlin ’19
3. Plabic Graphs Mago, Schreiber, Spradlin, Yelleshpur, AV ’20 ’21 He, Li ‘20
4. Tensor Diagrams Ren, Spradlin, AV ’21
5. Scattering Diagrams Herderschee ‘21

1. Tropical Geometry

- Speyer-Williams’03 associated a fan to the positive Grassmanian by solving tropicalized Plucker relations (multiplication->addition, addition->minimum).
- Building on this idea Drummond, Foster, Gurdogan, Kalousios’19 Henke, Papathanasiou’19 looked at a “smaller” version of Gr(4,8) fan by looking at particular Plucker coordinates.
- This fan has 272 rays that are g-vectors for cluster coordinates that include 180 rational n=8 letters.
- There are 2 exceptional rays from which they reproduced 18 algebraic n=8 letters.
- Henke and Papathanasiou’21 generalized this work and obtained n=9 letters.

2. Dual Polytopes

- Arkani-Hamed, Lam and Spradlin'19 looked at polytopes dual to these fans.
- To compute variables associated to the exceptional rays they used the method of Chang, Duan, Fraser, Li'19 and found evidence for the expected type of square roots.
- They conjectured these variables come from a generating function of the form

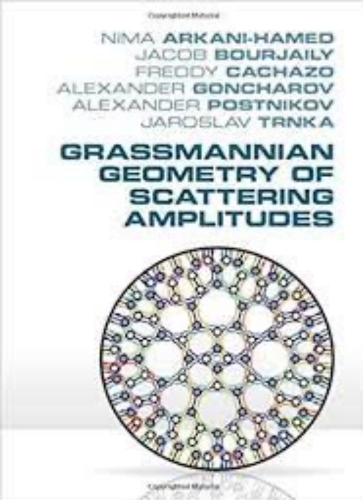
$$\frac{1}{1 - At + Bt^2}$$

$$A = \langle 1\,2\,5\,6\rangle\langle 3\,4\,7\,8\rangle - \langle 1\,2\,7\,8\rangle\langle 3\,4\,5\,6\rangle - \langle 1\,2\,3\,4\rangle\langle 5\,6\,7\,8\rangle$$

$$B = \langle 1\,2\,3\,4\rangle\langle 3\,4\,5\,6\rangle\langle 5\,6\,7\,8\rangle\langle 1\,2\,7\,8\rangle\,.$$

Poles at $A \pm \sqrt{A^2 - 4B}$

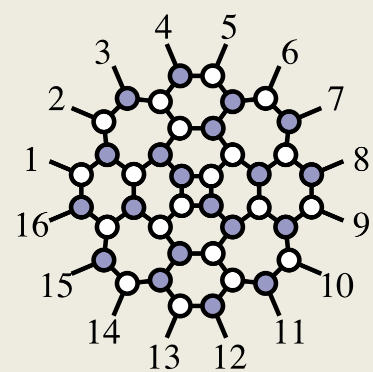
3. Plabic Graphs



The building blocks of N=4 SYM amplitudes are Yangian invariants which are given by integrals

$$\mathcal{Y}_{n,k}(\mathcal{Z}) = \frac{1}{\mathrm{vol}[\mathrm{GL}(k)]} \int \frac{d^{k\times n} C_{\alpha a}}{(1\cdots k)(2\cdots k{+}1)\cdots(n\cdots k{-}1)} \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a}\mathcal{Z}_a)$$

matrix C

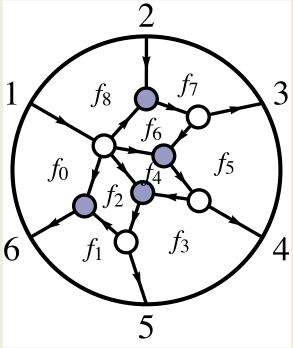


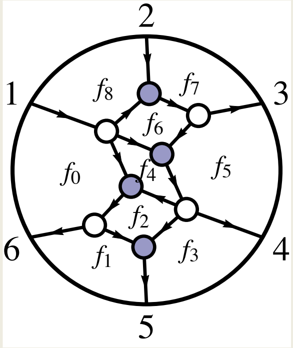
Our Strategy:

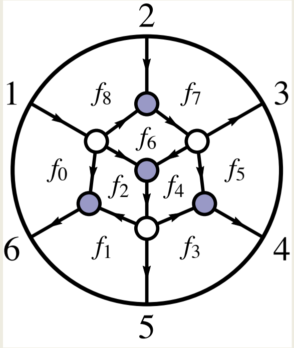
start with plabic graph, solve C Z=0, compare with known singularities

Mago, Schreiber, Spradlin, Yelleshpur Srikant AV'20 He, Li'20

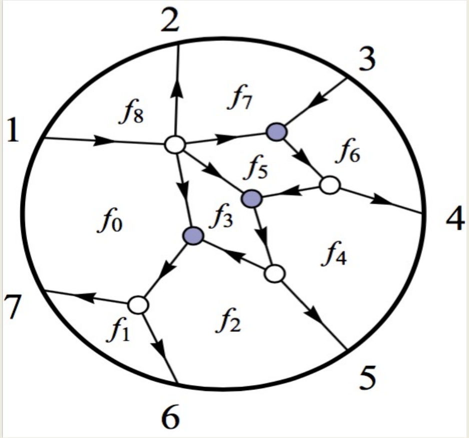
n=6 and n=7 letters

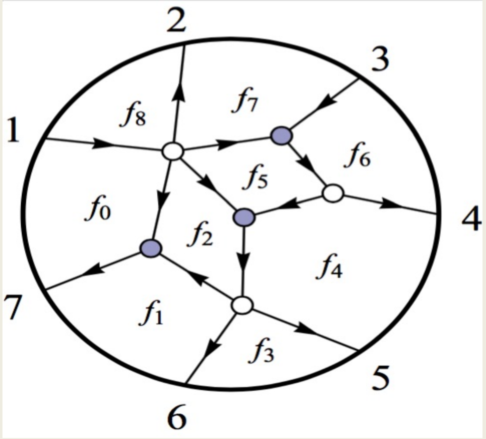






We exactly reproduce n=6 and n=7 singularities.

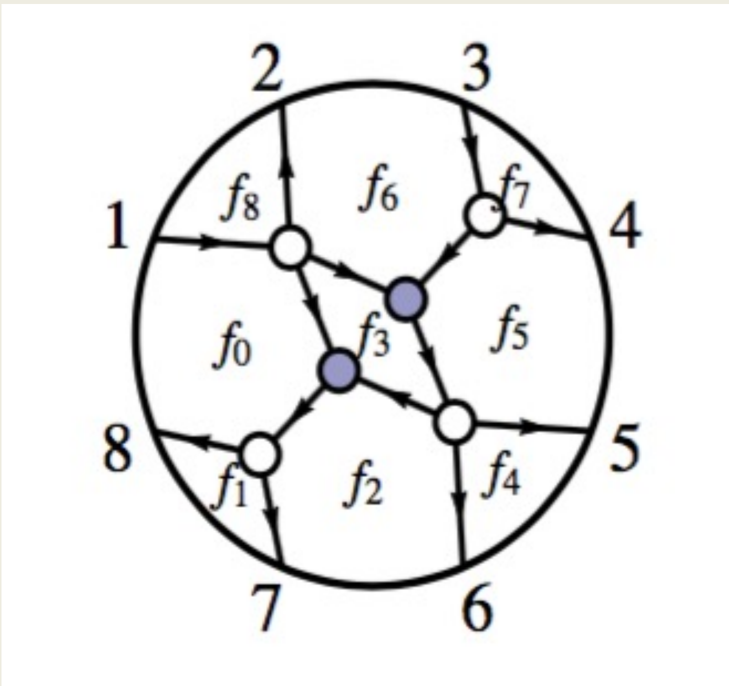


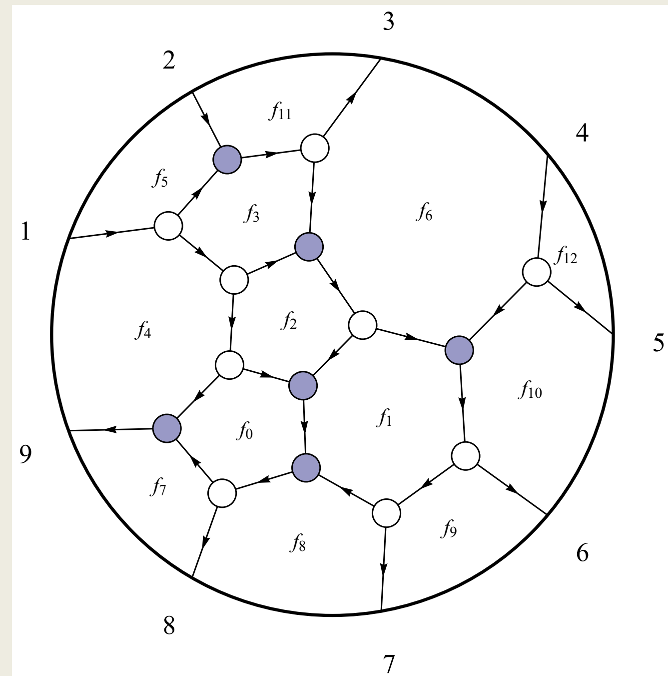


C Z=0

Algebraic letters

We can reproduce n=8 and n=9 algebraic letters.



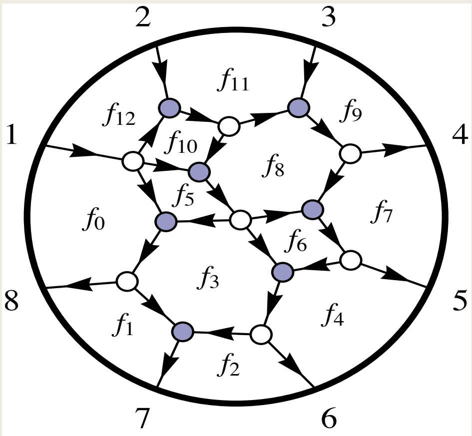


n=8:
this graph
plus square move on f3

n=9:
this graph
[mutations: more letters?]

Rational letters

- It is not possible to obtain all rational symbol letters from just plabic graphs. We have to consider non-plabic C-matrices.

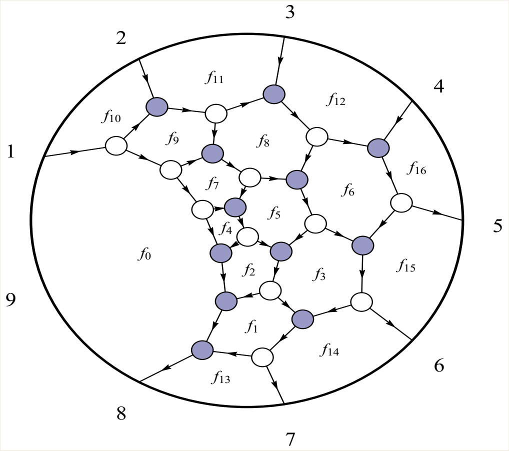


Mutation of face f_8 gives

non-plabic C'

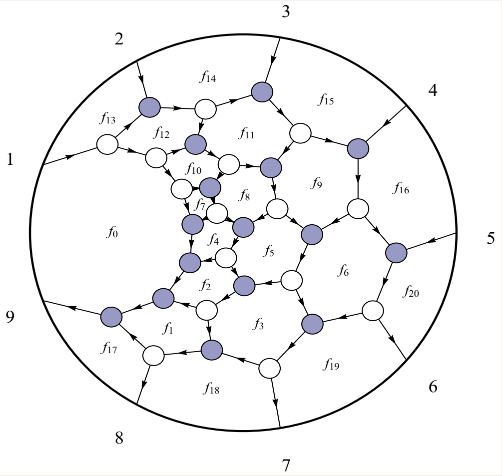
- Some solutions involve non-cluster coordinates.
- Restricting to the top cell (k=n-4) of the Grassmannian but allowing arbitrary non-plabic C-matrices, we will always produce cluster variables.

n=8 and n=9 rational letters



Starting with top cell, performing the following 13 mutation sequences, we can obtain extended n=8 alphabet (272 + 8 frozen):

$$\{\{4,7,8,3,6\},\ \{5,7,9,8,2\},\ \{5,8,3,1,2\},\ \{6,8,7,4,2\},\ \{7,1,2,5,6\},\ \{7,2,3,6,5\},\ \{7,4,2,3,6\},\ \{7,5,6,2,1\},\ \{8,3,5,2,4\},\ \{8,4,5,1,3\},\ \{8,6,3,2,4\},\ \{9,1,2,5,7\},\ \{9,8,5,3,1\}\}$$



Starting with top cell, performing the following 15 mutation sequences, we can obtain n=9 alphabet:

$$\{\{1,3,2,5,8,7,11,12\},\ \{1,5,2,10,8,10,12,11\},\ \{1,5,3,9,5,8,11,12\},\ \{2,4,6,5,9,8,11,9\},\ \{2,4,6,9,5,8,12,10\},\ \{2,4,7,8,11,8,12,10\},\ \{3,1,6,5,8,9,11,12\},\ \{3,4,2,5,8,4,7,10\},\ \{4,2,8,9,8,12,10,11\},\ \{5,6,3,7,11,10,8,12\},\ \{9,4,2,5,1,3,2\},\ \{9,11,6,4,8,7,10\},\ \{10,7,5,3,2,4,5\},\ \{11,6,3,2,4,7,10\},\ \{12,10,1,2,4,8,5\}\},$$

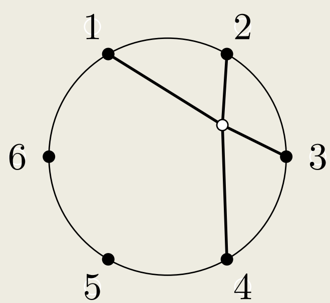
Symbol Alphabet from Plabic Graphs

- We identified set of graphs that reproduced all known n=8 and n=9 symbol alphabets.
- We do not have a theory to explain the pattern of which cells are associated to which symbol letter observed in amplitudes.
- We provided some “phenomenological” data in hope that future work will shed more light on this interesting problem.

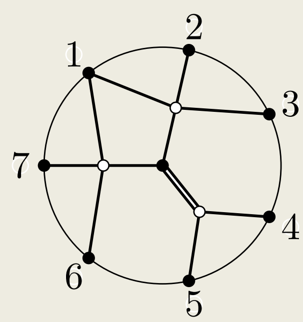
Mago, Schreiber, Spradlin, Yelleshput Srikant AV’20 ‘21

4. Tensor Diagrams

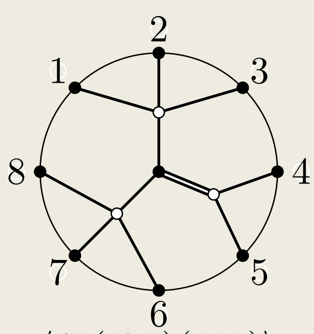
Cluster variables can be represented by tensor diagrams Fomin Pilyavsky’16



$\langle 1234 \rangle$



$\langle 1(23)(45)(67) \rangle$



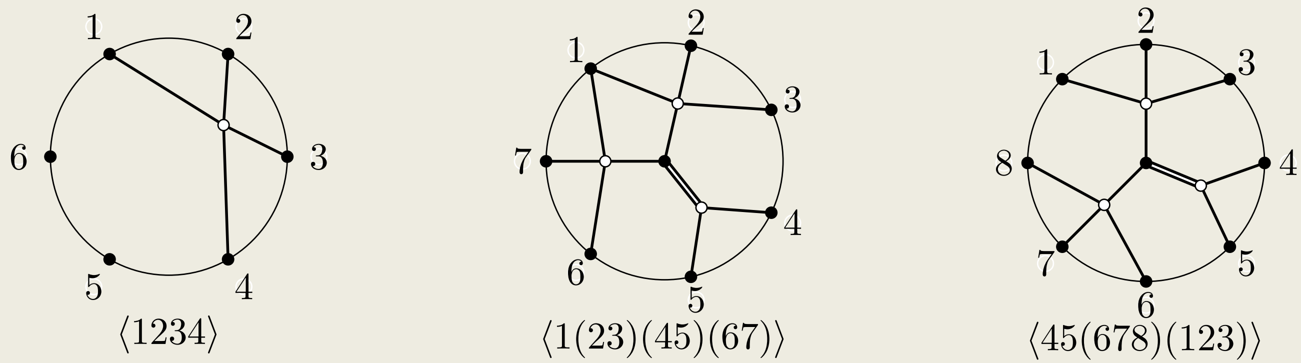
$\langle 45(678)(123) \rangle$

An n-point sl_k tensor diagram is a finite graph drawn inside a circle with n marked points along its boundary, satisfying

- all boundary vertices are colored black, and can have arbitrary valence
- each internal vertex may be black or white, but must have valence k
- each edge must connect a black and white vertex

4. Tensor Diagrams

Cluster variables can be represented by tensor diagrams Fomin Pilyavsky’16



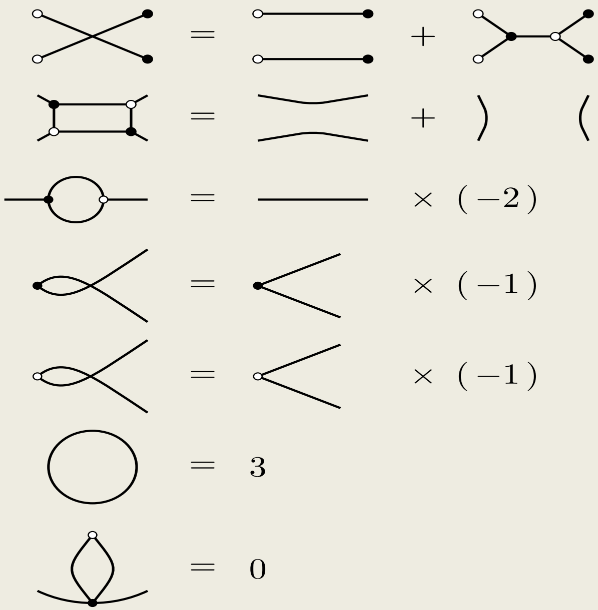
To each diagram D one associates an invariant $[D]$ by assigning

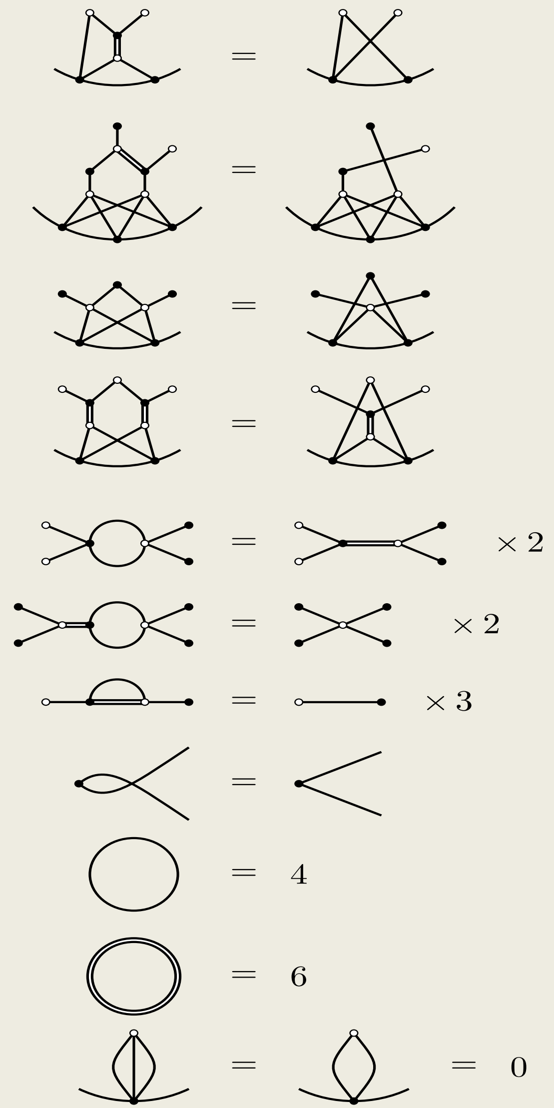
- a momentum twistor Z_i
- $\epsilon^{i_1 \cdots i_k}$ to each white vertex
- $\epsilon_{i_1 \cdots i_k}$ to each black vertex

and then contract the indices together as indicated by the edges.

Skein Relations

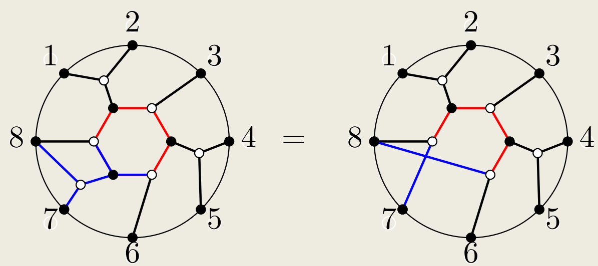
Tensor invariants [D] are invariant under graphical moves called skein relations.

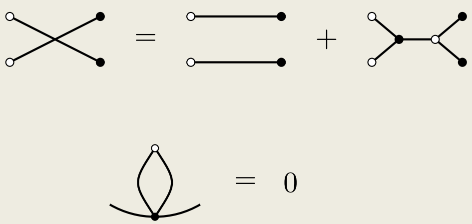




Fomin-Pylyavsky Conjecture

- A web is a planar tensor diagram.
- An aborizable web is a web that can be turned into a tree diagram using skein relations.





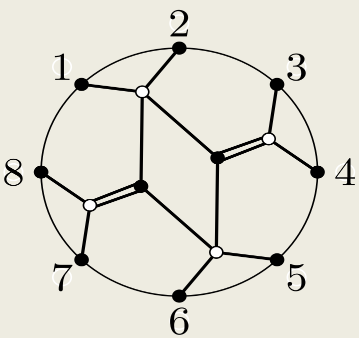
- Fomin-Pilyavsky ‘16 conjecture:

tensor invariants for an arborizable web are in one-to-one correspondence with cluster variables.

[Proven by Fraser ‘17 for Gr(3,9) and Gr(4,8).]

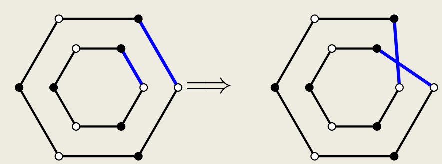
Algebraic Letters from Tensor Diagrams

- We proposed to look at almost aborizable webs (that can be reduced to having one inner loop), and assign to them a "web series"



$$\mathcal{W} = 1 + \sum_{m=1}^{\infty} t^m W_m$$

the coefficients can be derived graphically by twisting the inner loop



- We showed that the series takes the form:

$$\frac{1 - B\,t^2}{1 - A\,t + B\,t^2}$$

$$A = \langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle$$
$$B = \langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle\,.$$

- We observe square roots in the poles: $A \pm \sqrt{A^2 - 4B}$
- We reproduce square roots up to n=9.

Ren, Spradlin, AV'21

Conclusions

- Singularities of N=4 Yang-Mills amplitudes is described by Gr(4,n) cluster algebras for n=6, 7.
- Starting with n=8 one needs a mechanism producing finite subsets in Gr(4,n) and square roots.
- We studied candidate mechanisms coming from plabic graphs and tensor diagrams.
- Future: more systematics, more examples, cluster adjacency, cluster functions, non-N=4 SYM…..