Holographic duality for averaged WZW models

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Motivation

I will start by discussing some puzzles involving the sum over topologies in 3d gravity.

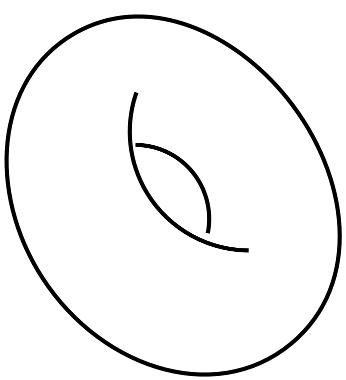
In 2007, Maloney and Witten attempted to calculate the torus partition function of pure gravity in three dimensions,

$$S = \frac{1}{16\pi G} \int \sqrt{g} \left(R + \frac{2}{\ell^2} \right)$$

They did this calculation by summing over saddles,

$$Z \overset{?}{=} \int Dg e^{-S}$$

$$\overset{?}{\approx} \sum_{SL(2,\mathbb{Z})} \exp\left[-\text{action of} \right.$$



$$\left. \right]$$

Poincaré Series

$$Z_{\text{3d gravity}}(\tau, \bar{\tau}) \stackrel{?}{=} \sum_{\text{SL(2,Z)}/\Gamma_\infty} \chi_0^{\text{Virasoro}}(\tau, \bar{\tau})$$

Vacuum character
(1-loop exact)

The result is inconsistent with AdS/CFT: Spectrum is **continuous** and **non-unitary**

[Maloney, Witten '07]
[Benjamin, Ooguri, Shao, Wang '19]

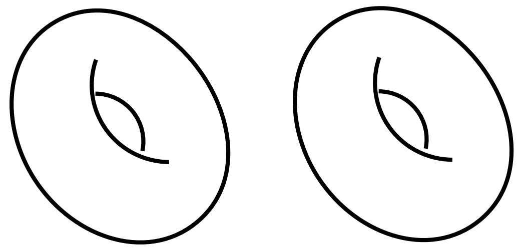
This is an avatar of the information puzzle: Gravitational path integrals do not reproduce the CFT microstates.

Factorization problem

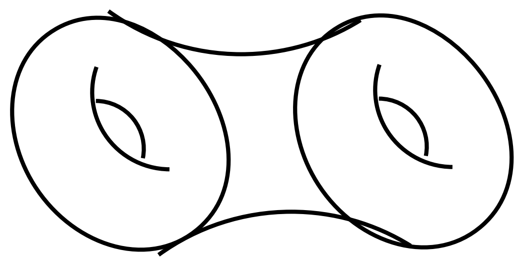
[Maldacena, Maoz; etc]

There is a similar problem if we consider multiple boundaries,

$$Z(\tau_1)Z(\tau_2) =$$



$+$



“double torus”

$$+ \quad \cdots$$

Again, the bulk path integral does not respect boundary quantum mechanics.

These are two distinct puzzles, but I think they probably need to be solved together.

See e.g. [Maxfield, Turiaci ’20]

3 points of view

The Skeptic

The sum over topologies is totally uncontrolled, let's ignore it.

No longer reasonable, after discovery of JT/Random matrix duality, replica wormholes, etc.

The Ensemble-Averager

Spacetime wormholes in the bulk mean that gravity is dual to an ensemble average of boundary CFTs

Maloney-Witten means the average spectrum is continuous.

True for JT gravity (but probably not in $D > 2$)

[Saad, Shenker, Stanford]

The Effective Field Theorist

The UV theory is quantum mechanics, but higher topologies do meaningfully compute certain (self-averaging) observables.

Presumably this is how it works in realistic UV-complete theories; but what observables are universal?

In this talk I will describe toy models for 3d gravity / 2d CFT where we can try to explore some of these issues.

Basic idea: replace $SL(2,\mathbb{R}) \to U(1)^N$

References

[Akfhami-Jeddi, Cohn, TH, Tajdini '20]

[Maloney, Witten '20]

[Dong, TH, Jiang '21]

See also: [Perez, Troncoso] [Cotler, Jensen] [Dymarsky, Shapere], [Datta et al], [Benjamin, Keller, Ooguri, Zadeh], [Ashwinkumar et al]

Narain Duality

[Afkhami-Jeddi, Cohn, TH, Tajdini '20]
and [Maloney, Witten '20]

Consider N free bosons in two dimensions.

This is a CFT with N^2 moduli.

Proposal: the ensemble average is holographically dual to an exotic theory of 3d gravity,

N free bosons averaged over moduli	$=$	$U(1)^N \times U(1)^N$ 3d Chern-Simons theory summed over topologies

"U(1) gravity"

Adding Matter to the Bulk

WZW models averaged over marginal deformations	$=$	$U(1)$ gravity plus topological matter

Why?

- $c > N_{\text{currents}}$
- correlators, factorization of subregions, ...

[Dong, TH, Jiang '21]

For another way to add matter, see [Benjamin, Keller, Ooguri, Zadeh '21]

The average torus partition function

Toroidal CFT: *N* compact bosons in 2D

$$Z(\tau,\bar{\tau};\Lambda) = \frac{1}{|\eta(\tau)|^{2N}}\Theta_\Lambda(\tau,\bar{\tau})$$

"Siegel-Narain theta"

$$\Lambda = \text{Narain lattice in } \mathbb{R}^{N,N}$$

Moduli space

Moduli space of Narain CFTs

$$\mathcal{M} \cong \frac{O(N,N)}{O(N)\times O(N)\times O(N,N,\mathbb{Z})}$$

Zamolodchikov metric = Haar measure for $O(N,N)$

Add "matter": Deformed WZW models

The $SU(N+1)_k$ WZW model has exactly marginal current-current deformations

$$S_{\rm WZW} \quad + \quad \int d^2 z J^i(z) \bar{J}^j(\bar{z}), \quad i,j \in \text{Cartan}$$

Bosonizing the Cartan currents gives the parafermion coset:

$$SU(N+1)_k = \left(\frac{SU(N+1)_k}{U(1)^N} \times \text{Narain}_\Lambda \right) / \Gamma_k$$

So the moduli space is *locally* equivalent to Narain moduli space. (Dualities differ).

The orbifold couples the boson sector to the parafermions,

$$Z = \frac{1}{|\eta(\tau)|^{2N}} \sum_{\alpha,\beta} Z^{\text{paraferm.}}_{\alpha,\beta} \Theta^{\alpha,\beta}_{\Lambda}$$

"Twisted Siegel-Narain theta"

[Förste, Roggenkamp '03] [Dong, TH, Jiang '21]

Average partition function:

$$\langle Z(\tau, \bar{\tau}) \rangle = \int dG dB \, Z(\tau, \bar{\tau}; \Lambda_{G,B})$$

For the free (untwisted) case, the average was calculated by C. Siegel in 1951!
Now understood as a special case of the Siegel-Weil formula,

$$\text{“} \quad (\text{Eisenstein}) = \int \Theta \quad \text{”}$$

Siegel calculated the average for any even lattice (not necessarily self-dual).

To state the theorem, let's first look at singularities of Z near the rationals,

$$\text{as} \quad \tau \to -\frac{d}{c} \; ,$$

$$\Theta_\Lambda(\tau, \bar{\tau}) \approx |c\tau + d|^{-N} R(c, d)$$

Coefficient depends on genus of lattice, but not on moduli

Siegel-Weil formula

$$\langle \Theta_\Lambda(\tau, \bar{\tau}) \rangle = \frac{1}{2} \sum_{(c,d)=1} |c\tau + d|^{-N} R(c, d)$$

For the genus of Narain lattices, *R(c,d)=1*, so

$$\langle \Theta_\Lambda(\tau, \bar{\tau}) \rangle = \frac{1}{2} \sum_{(c,d)=1} |c\tau + d|^{-N}$$

"Non-holomorphic Eisenstein series"

Turning on twists projects onto subset of terms in the sum,

$$\langle \Theta_\lambda^{(\alpha,\beta)} \rangle = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c\alpha + d\beta \in \Lambda}} |c\tau + d|^{-N}$$

Some intuition for Siegel-Weil

Cardy formula (plus other SL(2,Z) cusps) gives exact result for the *averaged* theory.

We know the Cardy formula is related to the BTZ black hole, so expect this to be related to a sum over bulk saddlepoints.

[Strominger '97] [Maldacena, Strominger '98] [Maloney, Witten '07]

Comparison to SUSY

Our Z is non-supersymmetric.

For a supersymmetric index, often the singularities at cusps can be used to reconstruct the whole function (Rademacher sums / "Farey tale")

[Dijkgraaf, Maldacena, Moore, Verlinde, Manschot]

For our non-holomorphic partition function, Siegel-Weil plays a similar role but it can only reconstruct the *average* over a "genus" of similar Z's.

Putting back in the eta functions, the average of the free boson is:

$$\langle Z \rangle = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} |\chi_0(\gamma\tau)|^2$$

where χ_0 is the vacuum character with $U(1)^N$ chiral algebra,

$$\chi_0(\tau) = \frac{1}{\eta(\tau)^N}$$

The sum can essentially be done. **Spectrum:**

- Continuous
- Extends down to the unitarity bound $\Delta \geq |\ell|$
- Vacuum state (compare: Liouville)

In the deformed WZW model, the average is

$$\langle Z \rangle = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,Z)} \sum_{\lambda} |c_0^\lambda(\gamma\tau)|^2$$

string function of affine algebra
($\sim$ parafermion character)

The average, in words:

Start with the WZW model.

Project onto the *uncharged U(1)* primaries.

Then sum over SL(2,Z).

This “project and sum over images” procedure will correspond to the bulk sum over topologies.

The holographic dual

$U(1)^N \times U(1)^N$ Chern-Simons

$$S_{\rm CS} = \sum_{i=1}^{N} \int_{M_3} \left(A_i dA_i - \tilde{A}_i d\tilde{A}_i \right)$$

summed over topologies (handlebodies).

“U(1) Gravity”

(Only has a semiclassical definition, but includes additional topologies + all loops)

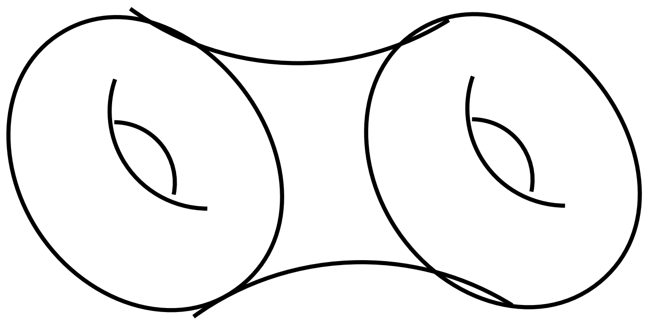
Adding matter

This is coupled to topological matter to account for the neutral parafermions.

$$\langle Z_{\mathrm{CFT}} \rangle = \sum_{\mathrm{solid\ tori}} Z_{\mathrm{bulk}}^{1-\mathrm{loop}}$$

Higher genus (without matter)

[Maloney, Witten '20]

$\sum$ 

$$= \int d(\text{moduli})\, Z(\tau_1, \bar{\tau}_1; \Lambda) Z(\tau_2, \bar{\tau}_2; \Lambda)$$

From this we can extract energy-level correlations

$$\overline{\rho_\ell(E)\rho_\ell(E')}_{\text{conn}} \quad \sim \quad \delta(E - E') + \text{non-zero corrections}$$

[Collier, Maloney '21][Cotler, Jensen '20]

Outlook

I do not think realistic theories of gravity have averaging.

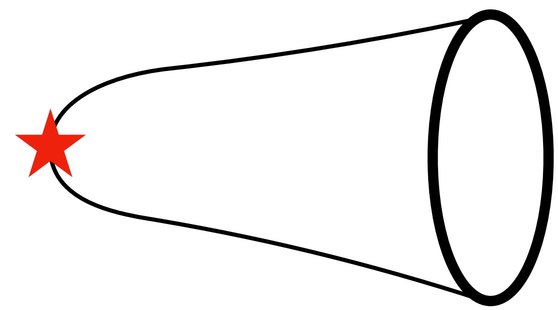
Nonetheless we can use ensemble averaging as a tool:

Coarse-grain over microscopic details and study universal, semi-microscopic aspects of gravity.

(*Cf.* RG, disorder in condensed matter)

Questions

What is the mechanism for averaging —> topology?



Which aspects of the microscopic theory can be extracted from averaging?

(*I.e.*, what observables are universal?)

Can the Narain duality be embedded into string theory?

Is there a role for the holomorphic Siegel-Weil formula in string theory?

Thank you