Exotic Theories and UV/IR Mixing

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IAS

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Gorantla, Lam, NS, Shao, 2108.00020 (and references therein)

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Introduction

Lore: the long-distance/low-energy behavior of every lattice system with short-range interactions is captured by a standard continuum quantum field theory.

Exotic lattice models, e.g., XY-plaquette model [Paramekanti, Balents, Fisher; …] and fracton models [Chamon; Haah; Vijay, Haah, and Fu; …] are notable counterexamples.

- Kinematics: exact or emergent exotic global symmetries (e.g., subsystem symmetries).
- Dynamics: UV/IR mixing, reminiscent of UV/IR mixing in certain string constructions (gravity, little string theory, field theory on a non-commutative space).

Modified Villain lattice versions of many systems [..., Gorantla, Lam, NS, Shao]

Differ from the original lattice formulations

- Exhibit properties of the corresponding continuum theory:
  - all global internal symmetries
  - ‘t Hooft anomalies between these symmetries
  - dualities
- Provide a rigorous formulation of the continuum theories.

We will not review or use them here – only use the fact that they exist.

Our goal here

Focus on one example.

Study the spectrum and some correlation functions.

The main result is various manifestations of UV/IR mixing.

Our discussion can be formulated on the lattice (in its modified Villain version) and then we can take the continuum limit.

Instead, we will present the analysis in the continuum, but we will see that occasionally we will need to restore the lattice.

An exotic theory: 2+1d ϕ-theory

[Paramekanti, Balents, Fisher; … NS, Shao; …]

$$S = \int d\tau dx dy \left(\frac{\mu_0}{2} (\partial_\tau \phi)^2 + \frac{1}{2\mu} \left(\partial_x \partial_y \phi\right)^2 \right) \qquad \phi \sim \phi + 2\pi$$

Subsystem global symmetries $\qquad \partial_\tau j_\tau = \partial_x \partial_y j_{xy}$

$$Q^x(x) = \oint dy\, j_\tau \qquad , \qquad Q^y(y) = \oint dx\, j_\tau$$

- $U(1)^m$ momentum $\qquad j_\tau^m = i\mu_0 \partial_\tau \phi\,, \qquad j_{xy}^m = \frac{i}{\mu} \partial_x \partial_y \phi$

  $\phi(x,y,\tau) \to \phi(x,y,\tau) + \alpha_x(x) + \alpha_y(y)$

  (α_x, α_y can be discontinuous)
- $U(1)^w$ winding $\qquad j_\tau^w = \frac{1}{2\pi} \partial_x \partial_y \phi\,, \quad j_{xy}^w = \frac{1}{2\pi} \partial_\tau \phi$
- Self-duality: $\phi \leftrightarrow \tilde{\phi}\,, \mu_0 \leftrightarrow \frac{\mu}{(2\pi)^2}\,, U(1)^m \leftrightarrow U(1)^w$

2+1d ϕ-theory – spectrum [NS, Shao]

For simplicity, $\ell_x = \ell_y = \ell$

$$\phi(x, y, t)$$

$$= \phi_x(x,t) + \phi_y(y,t) + \sum_{k_x, k_y \in \mathbb{Z}_{\neq 0}} a_{(k_x, k_y)}(t) e^{2\pi i \left(\frac{k_x x}{\ell} + \frac{k_y y}{\ell}\right)}$$

Plane waves (oscillators) with $\omega^2 = (2\pi)^4 \frac{k_x^2 k_y^2}{\mu \mu_0 \ell^4}$.

Because of this dispersion relation:

- $E \sim 1/\ell^2$ (and not $1/\ell$, as in more standard systems)
- For large ℓ, can have low E with large $p_x = k_x/\ell$, provided $p_y = k_y/\ell$ is sufficiently small – high momentum with low energy. This leads to UV/IR mixing. (More below.)

2+1d ϕ-theory – spectrum

$$\phi(x,y,t)$$
$$= \phi_x(x,t) + \phi_y(y,t) + \sum_{k_x,k_y\in\mathbb{Z}_{\neq 0}} a_{(k_x,k_y)}(t)\, e^{2\pi i\left(\frac{k_x x}{\ell}+\frac{k_y y}{\ell}\right)}$$

States charged under the momentum subsystem symmetry:

- The modes $\phi_x(x,t)$, $\phi_y(y,t)$ can be thought of as associated with the spontaneous breaking of this symmetry. In fact, the symmetry is restored in the quantum theory.
- They include the standard winding modes $\phi = \frac{2\pi}{\ell}\left(W_x x + W_y y\right)$ and hence these should not be considered separately.

2+1d ϕ-theory – spectrum

- For simplicity, ignore the common zero mode of $\phi_x(x,t)$ and $\phi_y(y,t)$. Then, $\phi_x(x,t)$ and $\phi_y(y,t)$ are independent rotors at different positions:

$$S = \frac{\ell\mu_0}{2}\int dt\left(\oint dx\,\left(\partial_t\phi_x(x,t)\right)^2 + \oint dy\left(\partial_t\phi_y(y,t)\right)^2\right)$$

Like 1+1d free fields without the spatial derivatives (pointwise periodic).

On the lattice with lattice spacing a, $\ell = aL$, and $\hat{x},\hat{y} = 1,2\ldots,L$,

$$H = \frac{1}{2\ell\mu_0\,a}\left(\sum_{\hat{x}} n_x(\hat{x})^2 + \sum_{\hat{y}} n_y(\hat{y})^2\right)\ ,\qquad n_x(\hat{x}), n_y(\hat{y}) \in \mathbb{Z}$$

Their energies diverge $E \sim \frac{1}{\mu_0\ell a} \to \infty$.

2+1d ϕ-theory – spectrum

What about states charged under the winding subsystem symmetry?

To be periodic modulo 2π and carry charge,

$$\phi = \frac{2\pi}{\ell}\left(x\Theta(y-y_0) + y\Theta(x-x_0) - \frac{xy}{\ell}\right) \qquad 0 \le x, y < \ell$$

$$j_\tau^w = \frac{1}{2\pi}\partial_x\partial_y\phi = \frac{1}{\ell}\left(\delta(y-y_0) + \delta(x-x_0) - \frac{1}{\ell}\right)$$

$$Q^x(x) = \oint dy\, j_\tau^w = \delta(x-x_0),$$

$$Q^y(y) = \oint dx\, j_\tau^w = \delta(y-y_0)$$

These configurations have infinite energy. Restoring the lattice spacing a, their energy is $\sim \frac{(2\pi)^2}{\mu\ell a} \to \infty$.

2+1d ϕ-theory – spectrum

To summarize:

- Plane waves (oscillators), created by $\partial_x\partial_y\phi$, $\partial_\tau\phi$, etc.

$$E \sim \frac{1}{\sqrt{\mu\mu_0}\,\ell^2}$$

- States charged under the momentum subsystem symmetry, created by $\exp(i\phi)$

$$E^m \sim \frac{1}{\mu_0 \ell a}$$

- States charged under the winding subsystem symmetry, created by $\exp(i\tilde{\phi})$

$$E^w \sim \frac{1}{\mu \ell a}$$

The momentum and winding states are exchanged by the self-duality.

Only the plane waves are present in the spectrum of the continuum theory.

2+1d ϕ-theory – spectrum

The main surprising result of the analysis of the spectrum is that the states charged under the momentum and winding subsystem symmetries have high energy – infinite in the continuum limit.

- The momentum and winding states exist in the Hilbert space of the lattice theory (in its modified Villain form), but they are not dynamical excitations in the continuum theory – they are not present in the Hilbert space of the continuum theory.
- Since they carry conserved charges, they are defects that can be added to the continuum theory. Hence, they are meaningful in the continuum limit. Note that they are exchanged by the self-duality.

2+1d ϕ-theory – UV/IR mixing

Go back to the lattice with $L_x = L_y = L$ sites and $\ell = aL$

Plane waves $\quad E \sim \frac{1}{\ell^2} = \frac{1}{L^2 a^2}$

Momentum and winding states $\quad E \sim \frac{1}{\ell a} = \frac{1}{L a^2}$

We are interested in $L \to \infty$.

Above, we took $a \to 0$ with fixed $\ell = La$. This kept the plane waves and pushed the charged states to infinity.

Alternatively, if we hold a fixed, i.e., $\ell \to \infty$, all these states approach zero energy.

We see that

$$[\ell \to \infty, a \to 0] \neq 0$$

UV/IR mixing.

2+1d ϕ-theory – correlation functions

Consider the lattice theory in its modified Villain form. Fix a gauge and then all the correlation functions are determined by the Green's function (propagator)

$$\langle\phi\phi\rangle = \text{explicit but complicated expression}$$

Study correlation functions of "good" local operators like $\exp(i\phi)$, $\Delta_\tau\phi$, etc. and then take $L_x = L_y = L \to \infty$. This can be done in two different ways

- Continuum limit: $a \to 0$, with fixed $\ell = aL$. Operators at fixed positions in space are separated by many (infinite in the limit) lattice sites. Can later take $\ell \to \infty$.
- Thermodynamic limit: fixed a, i.e., $\ell = aL \to \infty$. Can later separate operators to be many lattice spacings apart, i.e., $a \to 0$.

$\langle \partial_\tau \phi \, \partial_\tau \phi \rangle$

Take the continuum limit and then infinite volume – i.e., $\ell \to \infty$
(For simplicity, set $\mu = \mu_0 = 1$ and drop constants of order one.)

$$\langle \partial_\tau \phi(0,0,0) \partial_\tau \phi(x,y,\tau) \rangle \sim \begin{cases} -\dfrac{1}{(xy)^2} & |\tau| \ll |xy| \\ -\dfrac{1}{\tau^2} \log \dfrac{|\tau|}{|xy|} & |\tau| \gg |xy| \end{cases}$$

UV divergence as $xy \to 0$

- Finite on the lattice with nonzero a.
- As $x \to 0$ with fixed y, it is associated with large momenta p_x. (Similarly for $y \to 0$ with fixed x.)
- Because of the dispersion relation $\omega^2 \sim \left(p_x p_y\right)^2$, can have large p_x with finite ω, provided p_y is small enough.

$\langle \partial_\tau \phi \; \partial_\tau \phi \rangle$

$$\langle \partial_\tau \phi(0,0,0) \partial_\tau \phi(x,y,\tau) \rangle \sim \begin{cases} -\dfrac{1}{(xy)^2} & |\tau| \ll |xy| \\ -\dfrac{1}{\tau^2} \log \dfrac{|\tau|}{|xy|} & |\tau| \gg |xy| \end{cases}$$

Because of the dispersion relation $\omega^2 \sim \left(p_x p_y\right)^2$, we can have large p_x with finite ω, provided p_y is small enough.

Regularize the IR by restoring finite ℓ, then, $|p_y| \geq \frac{1}{\ell}$. The singularity as $x \to 0$ becomes $-\frac{1}{\tau^2} \log \frac{\ell}{|y|}$.

(If both x and y are small, it becomes $-\frac{1}{\tau^2} \log \frac{\ell^2}{|\tau|}$.)

This reflects the UV/IR mixing in the spectrum of plane waves.

$\langle \exp(i\phi) \exp(-i\phi) \rangle$ (momentum or winding)

The subsystem symmetry forces the two operators to be at the same spatial position (otherwise, the correlation function vanishes)

$$\langle \exp\big(i\phi(0,0,0)\big) \exp\big(-i\phi(0,0,\tau)\big) \rangle$$

As we take the continuum limit,

$$\langle \exp\big(i\phi(0,0,0)\big) \exp\big(-i\phi(0,0,\tau)\big) \rangle \sim \exp\left(-\frac{|\tau|}{\ell a}\right) \to 0$$

The exponent represents the energy of the lowest momentum state $\quad E \sim 1/\ell a \to \infty$.

These operators vanish in the continuum limit – they are infinitely irrelevant (redundant).

$\langle \exp(i\phi) \exp(-i\phi) \rangle$ (momentum or winding)

In the thermodynamic limit (finite a) [Paramekanti, Balents, Fisher]

$$\langle \exp\left(i\phi(0,0,0)\right) \exp\left(-i\phi(0,0,\tau)\right) \rangle \sim \exp\left(-\left(\log\left(\frac{|\tau|}{a^2}\right)\right)^2 \right)$$

For large τ, it decays faster than any power, but is not exponentially suppressed (as in the continuum limit). The a dependence cannot be absorbed in wave-function renormalization.

$\exp(i\phi)$ vanishes in the continuum limit – redundant operator

This reflects the UV/IR mixing in the spectrum of the momentum and winding modes – their energies go to zero as $L \to \infty$, but slower than the energies of the plane waves.

Many other models

- Gapped models with $\mathbb{Z}_N$ subsystem symmetries
- Gauge theories of subsystem symmetries
- More possible subsystem symmetries in 3+1d
- A certain 3+1d $\mathbb{Z}_N$ gauge theory of a subsystem symmetry describes the long-distance behavior of one of the most celebrated fracton models, the X-cube model [Vijay, Haah, Fu].

All these models have a modified Villain version and a corresponding continuum description.

They exhibit even more peculiar UV/IR mixing.

For example, the ground state degeneracy of the X-cube model depends on the number of sites:

$$N^{2(L_x+L_y+L_z)-3}$$

Summary

- The low-energy limit of a lattice theory is expected to be a continuum quantum field theory.
- Exotic lattice models are challenging counter-examples because
  - Subsystem global symmetry
  - UV/IR mixing
  - Large ground state degeneracy (infinite in the continuum limit)
  - Discontinuous and even singular observables in the continuum limit
  - Defects with restricted mobility
- Some peculiar continuum theories can capture these facts. They involve discontinuous fields. They can be made rigorous using modified Villain lattice models.

Thank you

Stay healthy