Recursion and doubling for scattering amplitudes: a geometric viewpoint

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Scattering amplitudes ...

- ... can be calculated in any reasonable quantum field theory or string theory.
- ... carry the symmetry and properties of a theory. They are perturbative objects.
- ... can be converted into cross sections (measurable).

Scattering amplitudes in QFT ...

are (usually) calculated using the Feynman formalism while

scattering amplitudes in string theory ...

are calculated as worldsheet correlators in the conformal worldsheet theory.

Taking the field theory limit of string amplitudes is cumbersome.

Scattering amplitudes in QFT

Feynman formalism

- Feynman formalism is believed to be predictive, amazingly well confirmed by experiment
- perturbative expansion $\rightarrow$ loop expansion
- integrals need to be regularized $\rightarrow$ usually use dimensional regularisation, obtain expansion in dimreg parameter ϵ
- regularization is very difficult and is usually done on an integral-by-integral basis.
- many structures understood for Feynman graphs:
  outer space (graph operations), tropical geometry, kinematical limits

Scattering amplitudes in string theory

Worldsheet formalism

- delivered as correlators on a suitable worldsheet.
- perturbative expansion $\rightarrow$ genus expansion
- expansion in α' (inverse string tension).
  point particle limit $\alpha' \rightarrow 0$ does not lead to Feynman representations
- can usually be split into a field theory part $+$ (scalar) string correction

Alternatives & shortcuts

- determine scattering amplitudes from *analytical constraints*
  soft limits, collinear limits, symmetry properties (e.g. dictated by color: photon-decoupling, BCJ-identities): bootstrap approaches, “symbol alphabet”

- includes transcendentality hierarchies, Galois theory etc.

- use Cutkowski rules, unitary cuts

Recursion

- in $\mathcal{N} = 4$ super-Yang-Mills theory: Parke-Taylor formula, BCFW recursion relations, CSW recursion relations. [Britto, Cachazo Feng, Witen] [Cachazo, Svrček Witten]

- more general for gauge theories: Berends-Giele (off-shell) recursion relations, Cutkowski rules [Berends Giele]

- geometrical pictures for *integrands*: amplituhedron & related [Arkani-Hamed et al.]

- string theory: Berends-Giele recursions [Mafra, Schlotterer Stieberger]

Doubling

- double-copy relations: relate sum of products of gauge amplitudes to gravitational amplitude [Bern, Carrasco Johansson]
- KLT relations: relate open to closed-string scattering amplitudes [Kawai, Lewellen Tye]
- gravitational waves: classical perturbative double-copy for wave-forms for inspirals/mergers
- some of the classical symmetries of closed-string/gravitational amplitudes can presumably e explainted from their “square structure”.

Current status

- recursion and doubling ease and enable many previously inaccessible calculations
- in no way, recursion and doubling are the prevalent tools: Feynman is omnipresent.
- there is some (very symmetric) scenarios, where recursions and doubling allow to determine *all* scattering amplitudes

Goal today

- discuss a versatile formalism modeling general *recursions*
- suggest a similar setup for *doubling*

A framework for recursions: requirements

Essential:

- need to model a recursion as a *smooth* process, ideally governed by an interpolating parameter
- use *regularization of endpoint divergencies* for a "uniformized" approach to regularization: understand regularization as a process inherently tied to the theory in question (and not a single integral)
- want recursion for *integrated expressions*, that is, complete scattering amplitudes

Nice to have:

- interpolating parameter should be connected to geometric interpretation
- need to have control over on-shell/off-shell recursion
- consider field theories as limits of string theories

How to model?

Take inspiration from field theory approach:

- Feynman integrals $\rightarrow$ Feynman parametrization $\rightarrow$ Symanzik polynomials
- Integration-by-parts identities: obtain "master integrals".
  Consider a whole class of integrals: *cohomology*.
- derive *differential equation* for master integrals
  mostly Fuchsian type, solve (e.g. with Frobenius method)
- calculate *fundamental group/monodromy group*
- express solutions as *iterated integrals*
  w.r.t. diffferentials transforming in the fundamental group

Similar to the setup for string amplitudes:

- master (iterated) integrals: *Selberg integrals*

$$\mathrm{S} = \mathrm{S}[](x_1, \ldots, x_L) = \prod_{0 \le x_i < x_j \le 1} \exp\left(s_{ij} \log x_{ji}\right) = \prod_{0 \le x_i < x_j \le 1} |x_{ij}|^{s_{ij}}$$

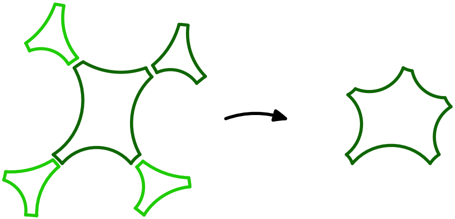
$$\mathrm{S}[i_{k+1}, \ldots, i_L](x_1, \ldots, x_k) = \int_0^{x_k} \frac{dx_{k+1}}{x_{k+1,i_{k+1}}} \mathrm{S}[i_{k+2}, \ldots, i_L](x_1, \ldots, x_{k+1})$$

- introduce an extra parameter point - interpolating between zero and one. This point is not integrated over - it is a parameter.
- the differential equation has poles at the end of the interpolating interval: determines iterated integrals: (elliptic) polylogarithms.
- recursion implemented in regularized limits of differential equation

Two examples (open string):

- recursion on tree-level: add/remove leg (Parke-Taylor for string theory)
- a recursion at one loop: from genus one to genus zero

Open-string recursion: add/remove leg



„model" the behaviour in terms of iterated integrals S_0 [Drinfeld]

differential equation for geometry parameter x_3:

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\mathbf{S}_0(x_3) = \left(\frac{e_0}{x_3} + \frac{e_1}{x_3 - 1}\right)\mathbf{S}_0(x_3)\,.$$

e_0, e_1: representations of generators of a (free) Lie algebra, matrices *linear* in parameters s_{ij}

$\Rightarrow$ ***Knizhnik–Zamolodchikov equation*** [Knizhnik Zamolodchikov]

Feynman integrals are hidden in limits of the geometry parameter x_3:

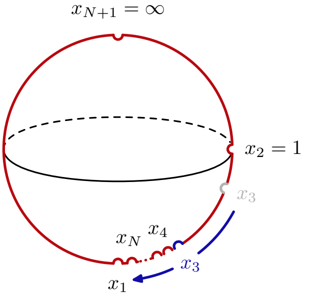
$x_3 \to 0$	$x_3 \to 1$

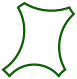
Open string recursion: add/remove leg

(Knizhnik–Zamolodchikov equation and Drinfeld associator)

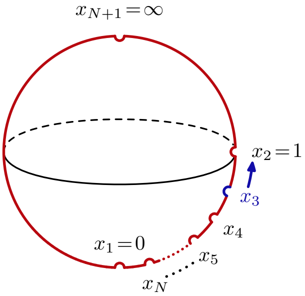
[Drinfeld]

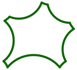
$$\mathbf{C}_0 = \lim_{x_3 \to 0} x_3^{-e_0}\, \mathbf{S}_0(x_3)$$



contains 

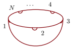
$$\mathbf{C}_1 = \lim_{x_3 \to 1} (1 - x_3)^{-e_1}\, \mathbf{S}_0(x_3)$$



contains 

$$\boxed{\Phi(e_0, e_1)\; \mathbf{C}_0 = \mathbf{C}_1 .}$$

Open string recursion: add/remove leg



Example: from three to four points

[Broedel, Schlotterer Stieberger, Terasoma][Broedel Kaderli]

$$\frac{d}{dx_3}\mathbf{S}_0(x_3) = \left(\frac{e_0}{x_3} + \frac{e_1}{x_3-1}\right)\mathbf{S}_0(x_3)\,,$$

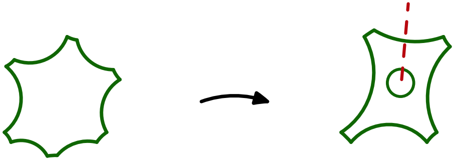
$$e_0 = \begin{pmatrix} s_{31}+s_{34} & -s_{34} \\ -s_{41} & s_{31}+s_{41} \end{pmatrix}, \qquad e_1 = \begin{pmatrix} s_{32} & 0 \\ s_{41} & s_{432} \end{pmatrix}$$

$$\begin{aligned}\Phi(e_0,e_1) = 1 &- \zeta_2[e_0,e_1] - \zeta_3[e_0+e_1,[e_0,e_1]] \\ &+ \zeta_4([e_1,[e_1,[e_1,e_0]]] + \tfrac{1}{4}[e_1,[e_0,[e_1,e_0]]] \\ &- [e_0,[e_0,[e_0,e_1]]] + \tfrac{5}{4}[e_0,e_1]^2) + \ldots \quad ,\end{aligned}$$

Effectively (after basis choice, rotation, taking the limits for Mandelstam variables):

$$\begin{aligned}(\mathbf{C}_1)_1 &= 1 - \zeta_2 s_{12}s_{23} + \zeta_3\, s_{12}s_{23}(s_{12}+s_{23}) - \zeta_4 s_{12}s_{23}\left(s_{12}^2 + \frac{1}{4}s_{12}s_{23} + s_{23}^2\right) + \cdots \\ &= \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}\end{aligned}$$

Open string recursion: cut a loop



„model" the behaviour in terms of iterated integrals S_1

differential equation for geometry parameter z_2: :

$$\frac{\mathrm{d}}{\mathrm{d}z_2}\mathbf{S}_1(z_2) = \sum_{n\geq 0} g_{21}^{(n)} e^{(n)}\,\mathbf{S}_1(z_2)\,,$$

$e^{(n)}$: representations of generators of a (free) Lie algebra

$\Rightarrow$ ***elliptic KZB equation*** [Bernard, Knizhnik Zamolodchikov]

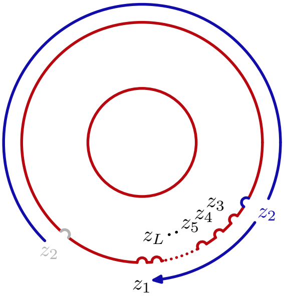
Feynman integrals are hidden in limits of the geometry parameter z_2:

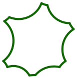
$z_2 \to 0$	$z_2 \to 1$

Open string recursion: cut a loop

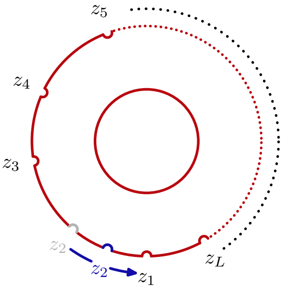
(Elliptic KZB equation and the KZB associator)

$$\mathbf{C}_0^{\mathsf{E}} = \lim_{z_2 \to 0} (2\pi i z_2)^{-e^{(1)}} \, \mathbf{S}_1(z_2)$$



contains 

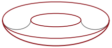
$$\mathbf{C}_1^{\mathsf{E}} = \lim_{z_2 \to 1} (2\pi i (1 - z_2))^{-e^{(1)}} \, \mathbf{S}_1(z_2) \,.$$



contains 

$$\boxed{\Phi^{\mathsf{E}}(e^{(0)}, e^{(1)}, \ldots) \; \mathbf{C}_0^{\mathsf{E}} = \mathbf{C}_1^{\mathsf{E}}}$$

Open string recursion: cut a loop



Example: from four-point tree-level to two-point one-loop

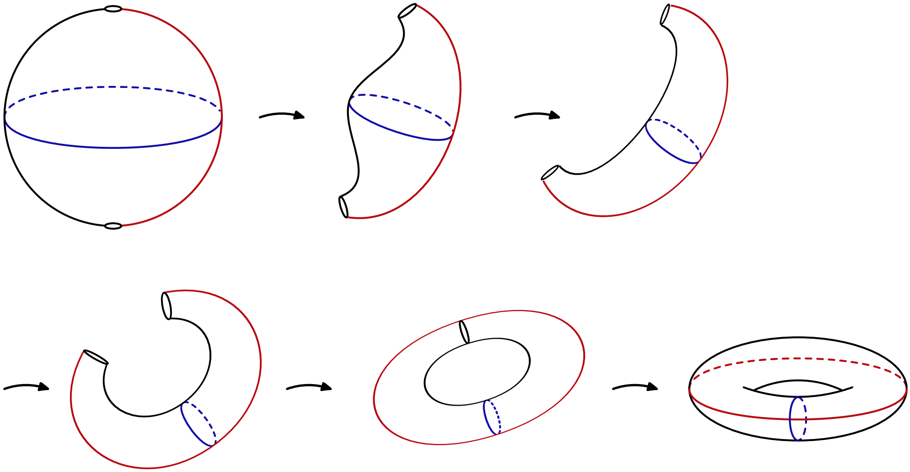
... involved, but just linear algebra.

$$S_{\text{2-point}}(\tilde{s}_{13}) = 1 + \tilde{s}_{13}^2 \left(\frac{1}{4}\zeta_2 - 3\gamma_0(4,0)\right) + \tilde{s}_{13}^3 \left(10\gamma_0(6,0,0) - 24\zeta_2\gamma_0(4,0,0) - \frac{1}{4}\zeta_3\right) + \tilde{s}_{13}^4 \Big(9\gamma_0(4,0,4,0) - 18\gamma(4,4,0,0) - 126\gamma_0(8,0,0,0) - \frac{3}{4}\zeta_2\gamma_0(4,0) - 144\zeta_4\gamma_0(4,0,0,0) + 240\zeta_2\gamma_0(6,0,0,0) + \frac{19}{64}\zeta_4\Big) + \mathcal{O}\Big((\alpha')^5\Big) \,.$$

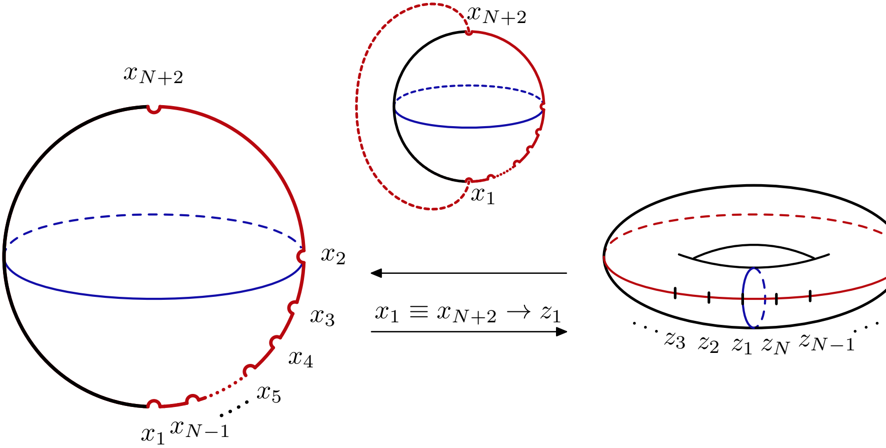
Reproduces all known one-loop results.

Instead: what happens to the torus geometrically?

Geometry of cutting/gluing:



Cutting of iterated integrals:

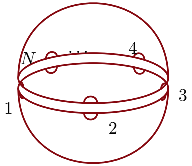


⇒ representation of torus degeneration in terms of iterated (Selberg) integrals

Can a similar setup be established for doubling?

e.g. open-closed string theory at genus zero?

- ***Yes:*** geometric idea is to blow up the real line to the Riemann sphere
- The geometric extra point interpolates between the (compactified) real line and the Riemann sphere.
- In order to ensure the second "copy" a combinatorial idea needs to be used in order to create the second set of iterated integrals.
- three fixed points: the (arbitrary) traversal of the other points will lead to the well-known expressions in terms of $(N-3)!$ integrals.
- Works for four points. General formalism not completely settled yet.



Back to field theory...

- there is several systems of QFT-amplitudes, where a recursion governed by an interpolating point and a differential equation should be possible to establish:
  - multi-Regge kinematics (single-valued polylogarithms)
  - Y-systems (single-valued multiple zeta values)

and string theory...

- find *recursion* for higher-genus string amplitudes
  cutting of several cycles, need Siegel modular forms, Arakelov green function ...
- use geometric setup to get a handle on a *doubling* relation at one loop

Outlook & further thoughts

- smooth geometric language very natural in string theory.
  theory in question $\rightarrow$ differential form on a suitable Riemann manifold.
- language is fundamentally different from any Feynman-based recursion relations/approaches
- *moduli spaces of Riemann spheres with marked points* and corresponding degenerations are well understood
  no structural obstacle for recursions on higher genera.
- immediate goal: find recursive setup for $\mathcal{N} = 4$ sYM theory, for which all known recursions show up as particular limits
- connect the algebraic approach to homotopy Lie algebras [Jurčo, Kim, Macrelli Saemann, Wolf]
- related to *topological recursion* [Borot Eynard]

Thanks!