BPS algebras & representations from colored crystals

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Reference

- Quiver Yangian from crystal melting
  [2003.08909] with *Masahito Yamazaki*

- Shifted quiver Yangians and representations from BPS crystals
  [2106.01230] with *Dmitrii Galakhov and Masahito Yamazaki*

- Toroidal and elliptic quiver BPS algebras and beyond
  [2108.10286] with *Dmitrii Galakhov and Masahito Yamazaki*

Main question

What is the algebraic structure underlying the BPS sector of a 4D $\mathcal{N} = 2$ gauge theory?

$$\boxed{\text{multiplication:} \qquad \mathcal{H}_{\text{BPS}} \otimes \mathcal{H}_{\text{BPS}} \to \mathcal{H}_{\text{BPS}}}$$

- Analogue of chiral algebra of 2D $\mathcal{N} = 2$ SCFT
- Robust
- Control many aspects of theory
  (BPS counting, wall crossing, etc)

History

- Fusion of two BPS states in 4D $\mathcal{N} = 2$ gauge theory

Harvey-Moore '96

$$\boxed{\text{multiplication:} \qquad \mathcal{H}_{\text{BPS}} \otimes \mathcal{H}_{\text{BPS}} \to \mathcal{H}_{\text{BPS}}}$$

  - Cohomological Hall Algebra (CoHA)

Kontsevich-Soibelman '10

- Include decaying process

$$\boxed{\text{co-multiplication:} \qquad \mathcal{H}_{\text{BPS}} \to \mathcal{H}_{\text{BPS}} \otimes \mathcal{H}_{\text{BPS}}}$$

  - Drinfeld double of CoHA $\longrightarrow$ affine Yangian algebra

Davison '13

- Known only for a few cases.

Setup

- Type IIA string in generic toric Calabi-Yau threefold X
- BPS states:

  D6/D4/D2/D0 branes wrapping holomorphic 6/4/2/0 cycles of X
- Question 1:

  What is the BPS algebra underlying this BPS sector?
- Question 2:

  What can we use BPS algebra for?

BPS quiver Yangians from colored crystals

1. BPS sector: $\mathcal{N}=4$ quiver QM (Q,W)

$$\Downarrow \quad \text{define}$$

2. { BPS states } = { colored crystals }

$$\text{act} \quad \Uparrow \Downarrow \quad \text{bootstrap}$$

3. BPS algebra = quiver Yangian $\mathsf{Y}(Q,W)$

BPS quiver Yangians from colored crystals

1. (Q, W) $\longleftarrow$ Input

$\Downarrow$ define

2. { BPS states } = { colored crystals }

act $\Uparrow \Downarrow$ bootstrap

3. quiver Yangian Y(Q, W) $\longleftarrow$ Output

Advantages

1. Explicit algebraic relations
2. Apply to ALL toric Calabi-Yau threefolds

Can be generalized to trigonometric and elliptic version

Representations

General representation of same quiver Yangian

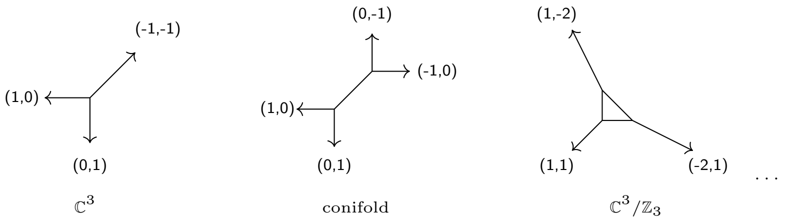
1. Given by subcrystals
2. Different framing of the same quiver
3. Describe other chambers, open BPS counting, and many more.

Outline

Setup

Type IIA string compactified on a toric Calabi-Yau threefolds X

- $X = \mathbb{T}^2 \times \mathbb{R}$ fibered overe $\mathbb{R}^3$



- 4D $\mathcal{N} = 2$ gauge theory
- $\frac{1}{2}$-BPS sector:

  D6/D4/D2/D0 branes wrapping holomorphic 6/4/2/0 cycles of X

$$\#(\text{D6, D4, D2, D0}) = (1, n_4^i, n_2^j, n_0)$$

What is the BPS algebra underlying this BPS sector?

$\longrightarrow$ How to describe these BPS states?

Low energy effective theory on D-brane bound state

$\mathcal{N} = 4$ quiver quantum mechanics (Q, W)

1. Quiver $Q = (Q_0, Q_1)$

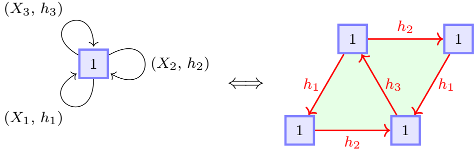
   $Q_0 = \{\text{vertex } a\}$ $\quad a: \quad U(N_a)$ gauge group

   $Q_1 = \{\text{arrow } I\}$ $\quad I \in \{a \to b\}: \;$ bi-fundamentals of $U(N_b) \times \overline{U(N_a)}$

2. superpotential $W = \sum$ monomials of Φ_I *Hanany et al*

$$\begin{cases} Q = (Q_0, Q_1) \\ W = \sum\limits_{L \in Q_2} \pm \mathrm{Tr}(\prod\limits_{I \in L} \Phi_I) \end{cases} \iff \text{periodic quiver } \tilde{Q} = (Q_0, Q_1, Q_2)$$

$\mathbb{C}^3$:



$W = \mathrm{Tr}[-X_1 X_2 X_3 + X_1 X_3 X_2]$

BPS crystal

Szendröi '07, Ooguri-Yamazaki '08

- 3D crystal description

  a D-brane bound state w/ #(D6, D4, D2, D0) = $(1, 0, n_2^j, n_0)$
  $\Longleftrightarrow$ a $U(1)^2$-invariant state of quiver QM (Q, W)
  $\Longleftrightarrow$ a 3D colored crystal configuration

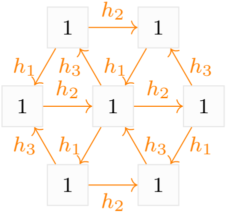
- Crystal generating function reproduces BPS partition function

$$Z_{\text{crystal}} = Z_{\text{BPS}}$$

-

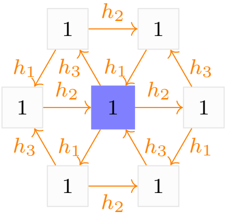
$$\text{periodic quiver} \quad \underset{\text{projection}}{\overset{\text{uplift}}{\rightleftharpoons}} \quad \text{full 3D crystal}$$

From periodic quiver to BPS crystal



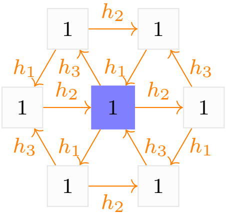
Origin of crystal

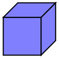
1. choose origin $\mathfrak{o}$



path ⇒ atom

1. choose origin $\mathfrak{o}$
2. path from $\mathfrak{o}$ ⇒ atom $\boxed{a}$





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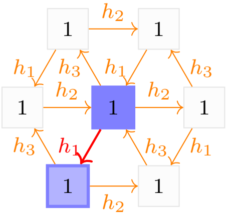

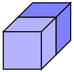

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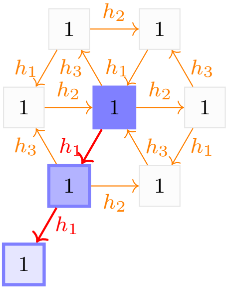
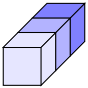
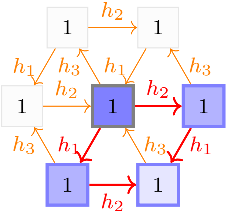
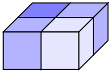




Path equivalence

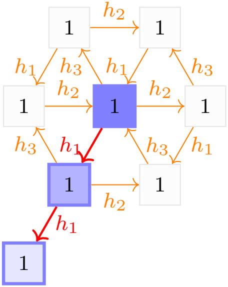
1. choose origin $\mathfrak{o}$
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. equivalence of paths

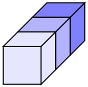




Melting rule

1. choose origin $\mathfrak{o}$
2. path from $\mathfrak{o}$ ⇒ atom $\boxed{a}$
3. equivalence of paths
4. Melting rule: if $\boxed{a} \notin \mathrm{K}$, then $I \cdot \boxed{a} \notin \mathrm{K}$





allowed

Melting rule

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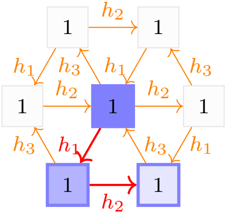

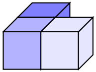

not allowed

Melting rule

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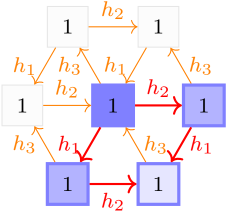
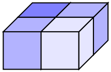
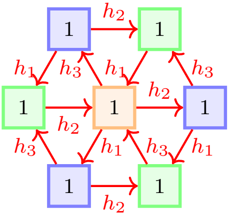


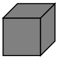


allowed

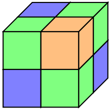
Depth of an atom

1. choose origin $\mathfrak{o}$
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. equivalence of paths
4. Melting rule: if $\boxed{a} \notin \mathrm{K}$, then $I \cdot \boxed{a} \notin \mathrm{K}$
5. depth = number of closed loop in the path



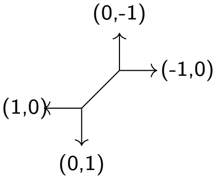


depth $= 0$

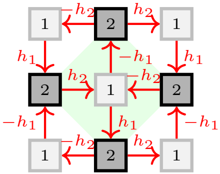


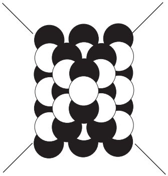
depth $= 1$

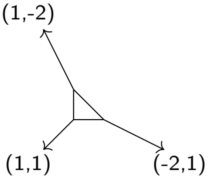
Toric CY_3 $\Longrightarrow$ periodic quiver $\Longrightarrow$ 3D crystal



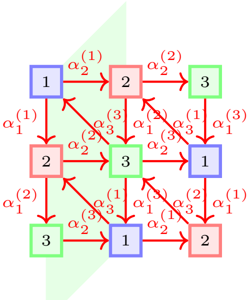
conifold

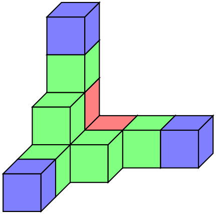






$\mathbb{C}^3/\mathbb{Z}_3$





Equivariant weight of arrows and atoms

1. choose origin $\mathfrak{o}$
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. equivalence of paths
4. Melting rule: if $\boxed{a} \notin \mathrm{K}$, then $I \cdot \boxed{a} \notin \mathrm{K}$
5. depth $=$ number of closed loop in the path

To derive BPS algebra from crystal, assign equivariant weights to atoms

L-Yamazaki '20

1. h_I: equivariant weight of arrow I
2. $h(\boxed{a})$: equivariant weight of atom $\boxed{a}$

Equivariant weight of arrows and atoms

1. choose origin $\mathfrak{o}$
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$ $\Longrightarrow$ $h(\boxed{a}) = \sum_{I \in \text{path}[\mathfrak{o} \to \boxed{a}]} h_I$
3. equivalence of paths $\Longrightarrow$ Loop constraint $\sum_{I \in L} h_I = 0$
4. Melting rule: if $\boxed{a} \notin \mathrm{K}$, then $I \cdot \boxed{a} \notin \mathrm{K}$
5. depth $=$ number of closed loop in the path

projection: same $h(\boxed{a})$ with different depth

To derive BPS algebra from crystal, assign equivariant weights to atoms

L-Yamazaki '20

1. h_I: equivariant weight of arrow I
2. $h(\boxed{a})$: equivariant weight of atom $\boxed{a}$

Number of equivariant parameters

- number of $h_I = |Q_1| (= |Q_0| + |Q_2|)$

$$(Q_0, Q_1, Q_2) = (\text{vertices}, \text{edges}, \text{faces})$$

- Loop constraints (global symmetry)

$$\sum_{I \in L} h_I = 0$$

- Vertex constraints (gauge symmetry)

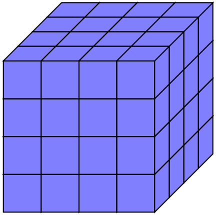
$$\sum_{I \in a} \text{sign}_a(I)\, h_I = 0$$

After loop and vertex constraints, the number of parameters = 2

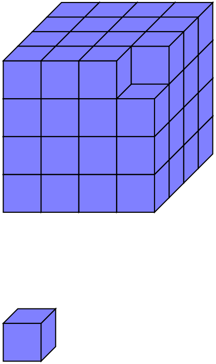
L-Yamazaki '20

Full crystal v.s. molten crystal ($\mathbb{C}^3$)

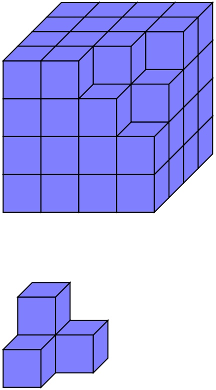
Vacuum



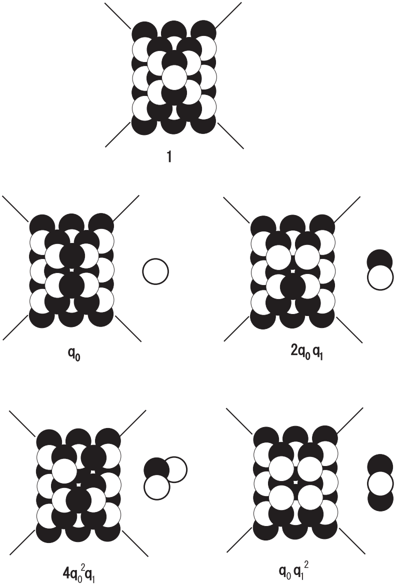
1-atom excited state



4-atom excited state



Full crystal v.s. molten crystal (resolved conifold)



Outline

Affine Yangian of $\mathfrak{gl}_1$

Associative algebra with generators e_j, f_j and ψ_j with $j \in \mathbb{Z}_0$

- Generators

$$\psi(z) = 1 + (h_1 h_2 h_3) \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}} \qquad e(z) = \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}} \qquad f(z) = \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}}$$

- One $\mathcal{S}_3$ invariant function $(h_1 + h_2 + h_3 = 0)$

$$\varphi_3(z) \equiv \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$$

- Defining relations

$$[\psi(z)\,,\psi(w)] \sim 0 \qquad [e(z)\,,f(w)] \sim -\frac{1}{\sigma_3}\,\frac{\psi(z)-\psi(w)}{z-w}$$

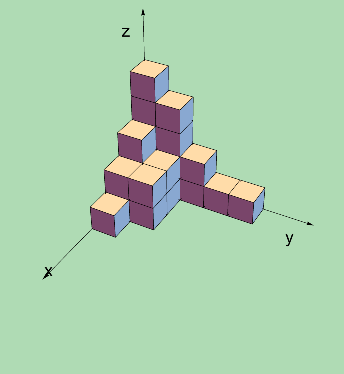
$$\psi(z)\,e(w) \sim \varphi_3(z-w)\,e(w)\,\psi(z) \qquad \psi(z)\,f(w) \sim \varphi_3^{-1}(z-w)\,f(w)\,\psi(z)$$

$$e(z)\,e(w) \sim \varphi_3(z-w)\,e(w)\,e(z) \qquad f(z)\,f(w) \sim \varphi_3^{-1}(z-w)\,f(w)\,f(z)$$

- Serre relations

$$\mathrm{Sym}_{(z_1,z_2,z_3)}(z_2-z_3)\,e(z_1)\,e(z_2)\,e(z_3) \sim \mathrm{Sym}_{(z_1,z_2,z_3)}(z_2-z_3)\,f(z_1)\,f(z_2)\,f(z_3) \sim 0$$

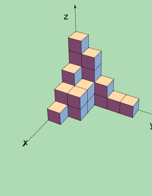
BPS crystal for $\mathbb{C}^3$: plane partitions



$$\sum_{n=0} M(n) q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k} = 1 + q + 3\, q^2 + 6\, q^3 + 13\, q^4 + 24\, q^5 + 48\, q^6 + \cdots$$

Action of $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$ on a plane partition

$$\begin{cases} \psi(z)|\Lambda\rangle = \Psi_\Lambda(z)|\Lambda\rangle\,, \qquad \Psi_\Lambda(z) \equiv \left(1+\frac{\psi_0\sigma_3}{z}\right)\prod_{\square\in(\Lambda)}\varphi_3(z-h(\square)) \\ e(z)|\Lambda\rangle = \sum_{\square\in\mathrm{Add}(\Lambda)} \frac{\sqrt{-\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)}}{z-h(\square)}|\Lambda+\square\rangle \\ f(z)|\Lambda\rangle = \sum_{\square\in\mathrm{Rem}(\Lambda)} \frac{\sqrt{+\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)}}{z-h(\square)}|\Lambda-\square\rangle \end{cases}$$



- Applying $e(z)$ on $|\emptyset\rangle$ repeatedly generates all $|\Lambda\rangle$ automatically
  Applying $f(z)$ on $\forall\ |\Lambda\rangle$ repeatedly brings it to $|\emptyset\rangle$.
- All poles of $\Psi_\Lambda(z)$ have a meaning: either Add(Λ) or Rem(Λ)
  $\Longrightarrow \Psi_\Lambda(z)$ only has poles near surface of $|\Lambda\rangle$.
- "Melting Rule" automatically satisfied !
  1. bonding factor $\varphi_3(z) \equiv \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$
  2. need loop constraint $h_1+h_2+h_3=0$

From affine Yangian of $\mathfrak{gl}_1$ to general quiver Yangian

1. Generators

$$\Big(e(z), \psi(z), f(z)\Big) \quad \Longrightarrow \quad \Big(e^{(a)}(z), \psi^{(a)}(z), f^{(a)}(z)\Big) \text{ for each } a \in Q_0$$

2. Algebraic relations?

Fix action of quiver Yangian on set of colored crystals first.

Action of quiver Yangian on colored crystal

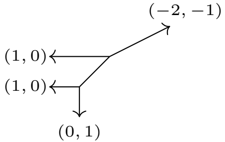
L-Yamazaki '20

$$\begin{cases} \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \,, \qquad \Psi^{(a)}_{\mathrm{K}}(u) \equiv \left(\frac{1}{z}\right)^{\delta_{a,1}} \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{a\Leftarrow b}(u - h(\boxed{b})) \\ e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Add(K)}} \frac{\pm\sqrt{\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \,, \\ f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Rem(K)}} \frac{\pm\sqrt{(-1)^{|a|}\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \,, \end{cases}$$

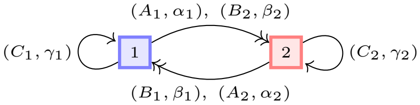
- Applying $e^{(a)}(z)$ on $|\emptyset\rangle$ repeatedly generates all $|\mathrm{K}\rangle$
  Applying $f^{(a)}(z)$ on $\forall\ |\mathrm{K}\rangle$ repeatedly brings it to $|\emptyset\rangle$.
- All poles of $\Psi^{(a)}_{\mathrm{K}}(z)$ have meaning: either Add$^{(a)}$(K) or Rem$^{(a)}$(K)
  $\Longrightarrow \Psi^{(a)}_{\mathrm{K}}(z)$ only has poles near surface of $|\mathrm{K}\rangle$.
- "Melting Rule" automatically satisfied !
  1. bonding factor $\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}} \frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)}$
  2. loop constraint $\sum_{I\in \text{loop } L} h_I = 0$

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$

- From toric data to quiver data



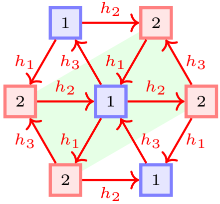
quiver and superpotential



$$W = \mathrm{Tr}[-C_1\, A_1\, B_1 + C_1\, B_2\, A_2 - C_2\, A_2\, B_2 + C_2\, B_1\, A_1]$$

$\Longleftrightarrow$

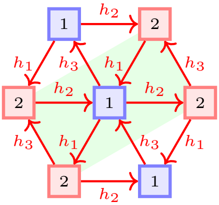
periodic quiver

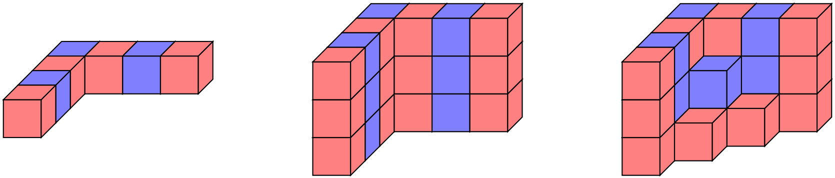


- After loop and vertex constraints $h_1 + h_2 + h_3 = 0$
- building blocks of $\Psi^{(a)}_{\mathrm{K}}(u)$

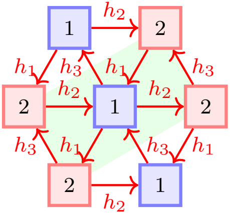
$$\varphi^{1\Leftarrow 1}(u) = \varphi^{2\Leftarrow 2}(u) = \frac{u+h_3}{u-h_3} \qquad \text{and} \qquad \varphi^{1\Leftarrow 2}(u) = \varphi^{2\Leftarrow 1}(u) = \frac{(u+h_1)(u+h_2)}{(u-h_1)(u-h_2)}$$

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: 2-colored plane partitions





$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal

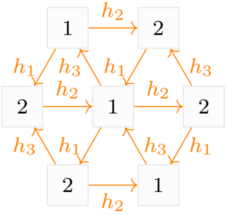


$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: vacuum

vacuum $|\emptyset\rangle$

1. Charge functions

$$\begin{cases} \Psi^{(1)}_{\mathrm{K}}(z) = \dfrac{1}{z} \\ \Psi^{(2)}_{\mathrm{K}}(z) = 1 \end{cases}$$

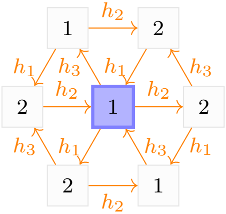


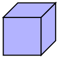
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: vacuum $\Longrightarrow$ level-1

vacuum $|\emptyset\rangle$

1. Charge functions
$$\begin{cases}\Psi^{(1)}_{\mathrm{K}}(z)=\frac{1}{z}\\ \Psi^{(2)}_{\mathrm{K}}(z)=1\end{cases}$$
2. Pole for $\boxed{1}$: $z=0 \quad\Longrightarrow\quad e^{(1)}(z)|\emptyset\rangle=\frac{\#}{z}|\boxed{1}\rangle$

   Pole for $\boxed{2}$: none $\quad\Longrightarrow\quad e^{(2)}(z)|\emptyset\rangle=0$
3. $f^{(1)}(z)|\emptyset\rangle=f^{(2)}(z)|\emptyset\rangle=0$



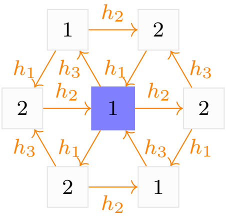


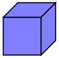
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-1

1-atom state $|\boxed{1}\rangle \quad \Longrightarrow \quad h(\boxed{1})=0$

1. Charge functions

$$\begin{cases} \psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1})) = \dfrac{1}{z}\cdot\dfrac{z+h_3}{z-h_3} \\ \psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow 2}(z-h(\boxed{1})) = \dfrac{(z+h_1)(z+h_2)}{(z-h_1)(z-h_2)} \end{cases}$$





$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-1 $\Longrightarrow$ level-2

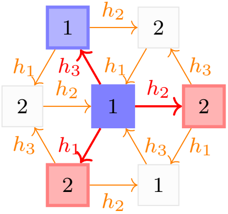
1-atom state $|\boxed{1}\rangle \quad \Longrightarrow \quad h(\boxed{1})=0$

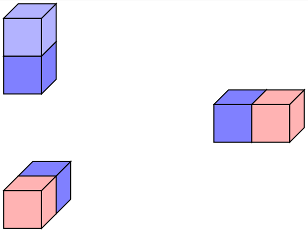
1. Charge functions

$$\begin{cases} \psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1})) = \dfrac{1}{z}\cdot\dfrac{z+h_3}{z-h_3} \\ \psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow 2}(z-h(\boxed{1})) = \dfrac{(z+h_1)(z+h_2)}{(z-h_1)(z-h_2)} \end{cases}$$

2. Pole for $\boxed{1}$: $z=0$ and $z=h_3$
   Pole for $\boxed{2}$: $z=h_1$ and $z=h_2$

3. $f^{(1)}(z)|\boxed{1}\rangle = |\emptyset\rangle$ and $f^{(2)}(z)|\boxed{1}\rangle = 0$



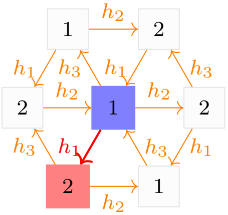


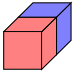
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-2

2-atoms state $|\boxed{1}_0\boxed{2}_1\rangle \quad \Longrightarrow \quad h(\boxed{1}_0)=0,\ h(\boxed{2}_1)=h_1$

1. Charge function

$$
\begin{cases}
\Psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 1}(z-h(\boxed{2})) \\
\qquad = \dfrac{(1)}{\cancel{z}}\cdot\dfrac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\dfrac{\cancel{(z)}(z+h_2-h_1)}{(z-2h_1)\cancel{(z+h_3)}} \\
\Psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow 2}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 2}(z-h(\boxed{2})) \\
\qquad = \dfrac{(z+h_1)\cancel{(z+h_2)}}{(z-h_1)(z-h_2)}\cdot\dfrac{(z+h_3-h_1)}{\cancel{(z+h_2)}}
\end{cases}
$$



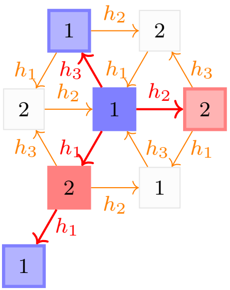


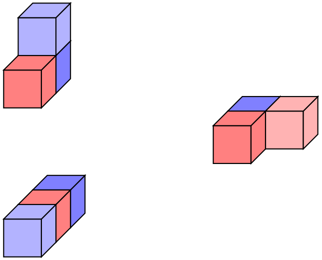
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-2 $\Longrightarrow$ level-3

2-atoms state $|\boxed{1}_0\boxed{2}_1\rangle \quad\Longrightarrow\quad h(\boxed{1}_0)=0,\ h(\boxed{2}_1)=h_1$

1. Charge function

$$\begin{cases} \Psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 1}(z-h(\boxed{2})) \\ \quad = \dfrac{(1)}{\cancel{z}}\cdot\dfrac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\dfrac{\cancel{(z)}(z+h_2-h_1)}{(z-2h_1)\cancel{(z+h_3)}} \\ \Psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow 2}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 2}(z-h(\boxed{2})) \\ \quad = \dfrac{(z+h_1)\cancel{(z+h_2)}}{(z-h_1)(z-h_2)}\cdot\dfrac{(z+h_3-h_1)}{\cancel{(z+h_2)}} \end{cases}$$



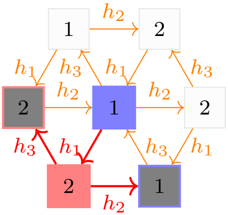


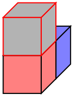
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: Melting Rule

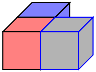
2-atoms state $|\boxed{1}_0\boxed{2}_1\rangle \quad \Longrightarrow \quad h(\boxed{1}_0)=0,\ h(\boxed{2}_1)=h_1$

1. Charge function

$$
\begin{cases}
\Psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 1}(z-h(\boxed{2})) \\
\qquad = \dfrac{(1)}{\cancel{z}}\cdot\dfrac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\dfrac{\cancel{(z)}(z+h_2-h_1)}{(z-2h_1)\cancel{(z+h_3)}} \\
\Psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow 2}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 2}(z-h(\boxed{2})) \\
\qquad = \dfrac{(z+h_1)\cancel{(z+h_2)}}{(z-h_1)(z-h_2)}\cdot\dfrac{(z+h_3-h_1)}{\cancel{(z+h_2)}}
\end{cases}
$$





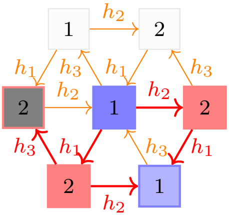


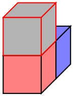
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: Melting Rule

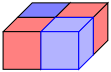
3-atoms state $|\boxed{1}_0\boxed{2}_1\boxed{2}_2\rangle \implies h(\boxed{1}_0)=0,\ h(\boxed{2}_1)=h_1\ \ h(\boxed{2}_2)=h_2$

1. Charge function

$$\begin{cases}\Psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow1}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow1}(z-h(\boxed{2}_1))\cdot\varphi^{2\Rightarrow1}(z-h(\boxed{2}_2)) \\ \qquad = \frac{(1)}{\cancel{z}}\cdot\frac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\frac{\cancel{(z)}(z+h_2-h_1)}{(z-2h_1)\cancel{(z+h_3)}}\cdot\frac{(z)(z+h_1-h_2)}{(z-2h_2)(z+h_3)} \\ \Psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow2}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow2}(z-h(\boxed{2}_1))\cdot\varphi^{2\Rightarrow2}(z-h(\boxed{2}_2)) \\ \qquad = \frac{\cancel{(z+h_1)}\cancel{(z+h_2)}}{(z-h_1)(z-h_2)}\cdot\frac{(z+h_3-h_1)}{\cancel{(z+h_2)}}\cdot\frac{(z+h_3-h_2)}{\cancel{(z+h_1)}}\end{cases}$$







Poles of $\Psi^{(a)}_{\mathrm{K}}(z)$ encode the positions of $\boxed{a} \in \mathrm{Add}(\mathrm{K})$ and $\mathrm{Rem}(\mathrm{K})$

1. Each $\boxed{b}$ in K contributes a factor of $\varphi^{a\Leftarrow b}(z - h(\boxed{b}))$ to $\Psi^{(a)}_{\mathrm{K}}(z)$
2. $h(\boxed{b}) \equiv \sum_{I \in \mathrm{path}[\mathfrak{o} \to \boxed{b}]} h_I$
3. $$\boxed{\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}} \frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)}}$$
4. loop constraint $\sum_{I\in \mathrm{loop}\ L} h_I = 0$

Poles are always pushed to the surface of crystal !

"Melting rule" is automatically implemented

Deriving algebra from its action on crystal representation

L-Yamazaki '20

$$\begin{cases} \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle\,, \qquad \Psi^{(a)}_{\mathrm{K}}(u) \equiv \left(\frac{1}{z}\right)^{\delta_{a,1}} \prod_{b\in Q_0}\prod_{\boxed{b}\in \mathrm{K}} \varphi^{a\Leftarrow b}(u-h(\boxed{b})) \\ e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Add}(\mathrm{K})} \frac{\pm\sqrt{\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle\,, \\ f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Rem}(\mathrm{K})} \frac{\pm\sqrt{(-1)^{|a|}\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle\,, \end{cases}$$

$$\begin{aligned} \psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,, \\ \psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\,, \\ e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,e^{(a)}(z)\,, \\ \psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,, \\ f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,, \\ \left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,, \end{aligned}$$

Relations in terms of modes

To compare to other algebras, convert to relations in terms of modes

1. Read off mode expansion of $(e^{(a)}(z), \psi^{(a)}(z), f^{(a)}(z))$ from algebra's action on crystals
2. Plug in mode expansions to algebraic relations and take singular terms:

$$\left[\psi_n^{(a)}\, ,\, \psi_m^{(b)}\right] = 0\, ,$$

$$\sum_{k=0}^{|b\to a|} (-1)^{|b\to a|-k}\, \sigma^{b\to a}_{|b\to a|-k}\, [\psi_n^{(a)}\, e_m^{(b)}]_k = \sum_{k=0}^{|a\to b|} \sigma^{a\to b}_{|a\to b|-k}\, [e_m^{(b)}\, \psi_n^{(a)}]^k\, ,$$

$$\ldots$$

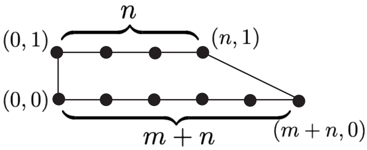
$$\left[e_n^{(a)}\, ,\, f_m^{(b)}\right\} = \delta^{a,b}\, \psi^{(a)}_{n+m}\, ,$$

$$[A_n\, B_m]_k \equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} A_{n+k-j}\, B_{m+j}\, , \quad [B_m\, A_n]^k \equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} B_{m+j}\, A_{n+k-j}\, .$$

$\sigma_k^{a\to b} \equiv k^{\text{th}}$ elementary symmetric sum of the set $\{h_I\}$ with $I \in \{a \to b\}$

Compare with known algebras

- Toric CY_3 $xy = z^m w^n$



$$\text{quiver Yangian} \quad = \quad \text{affine Yangian of } \mathfrak{gl}_{m|n}$$

Ueda '19

- For general toric CY_3, quiver Yangian is new algebra

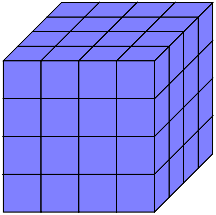
Derivation of quiver Yangians

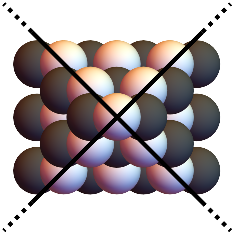
- bootstrapped from action on crystals *L-Yamazaki '20*
- confirmed from $\mathcal{N} = 4$ quiver quantum mechanics
  Galakhov-Yamazaki '20

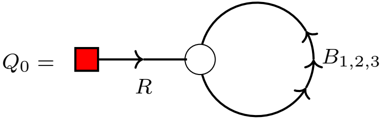
Outline

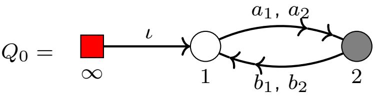
So far: canonical crystal

$\mathbb{C}^3$ | resolved conifold



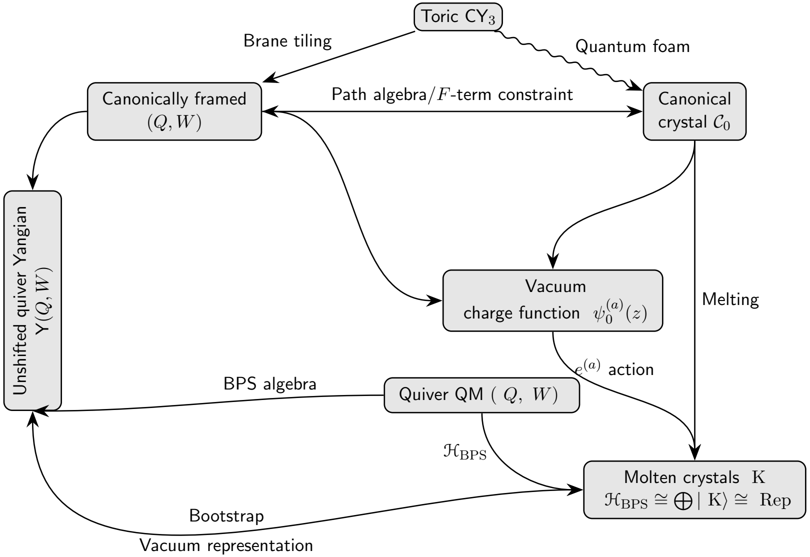






vacuum charge function $\psi_0^{(a)}(z) = \left(\frac{1}{z}\right)^{\delta_{a,\mathfrak{o}}}$

Canonical crystal, quiver Yangians, and quiver QM

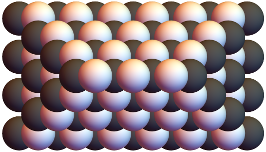


From canonical crystal to other crystals

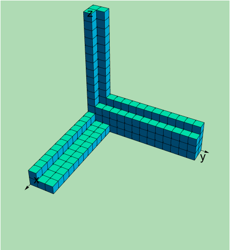
- The canonical crystal corresponds to counting of closed BPS invariants in the non-commutative DT chamber.

- Can have crystal with other shapes

wall crossing to other chambers



Open BPS states



- Can consider arbitrary subcrystals of canonical crystal

Subscrystal $^\sharp\mathcal{C}$

1. How to describe an arbitrary subcrystal $^\sharp\mathcal{C}$?

2. What is their relations to quiver Yangian?

3. What is their relation to the quiver?

Subscrystal $^\sharp\mathcal{C}$

1. How to describe an arbitrary subcrystal $^\sharp\mathcal{C}$?

   $\Longrightarrow$ superposition of positive/negative canonical crystals

2. What is their relations to quiver Yangian?

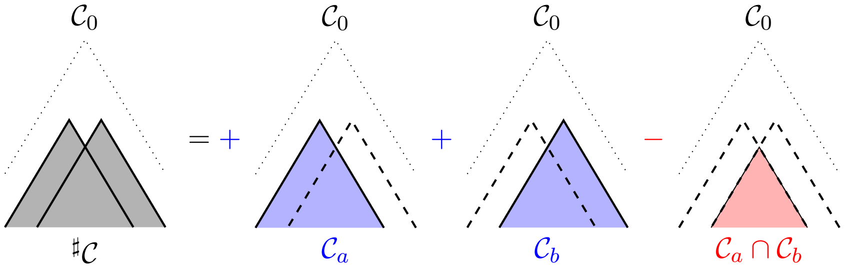
   $\Longrightarrow$ non-vacuum representations of (shifted) quiver Yangians

3. What is their relation to the quiver?

   $\Longrightarrow$ different framing of the original quiver

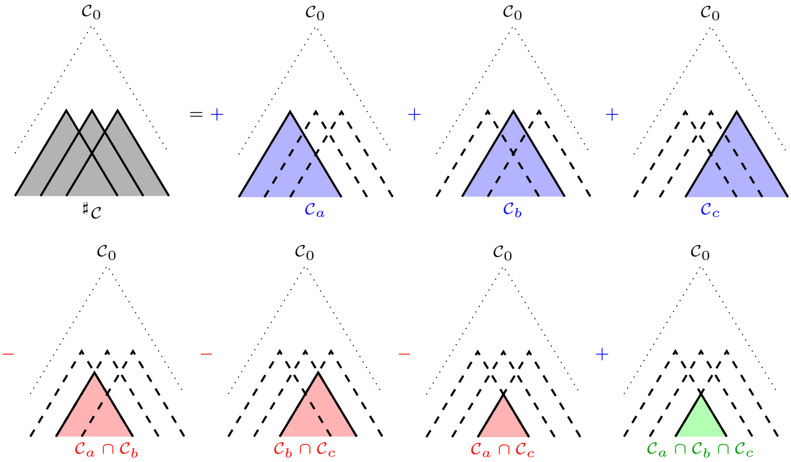
Decomposing subcrystal $^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

- step-1: determine the positions of positive crystals
- step-2: determine the overlaps of positive crystals
  $\Longrightarrow$ add negative crystals to cancel the overlaps



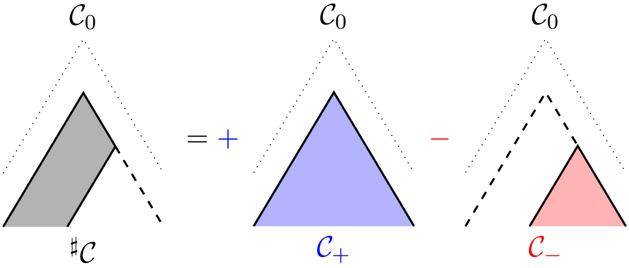
- step-3: determine the overlaps of negative crystals
  $\Longrightarrow$ add positive crystals to cancel overlaps of negative crystals

- step-4: continue until $^{\sharp}\mathcal{C}$ is reproduced (inclusion-exclusion principle)



Decomposing subcrystal $^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

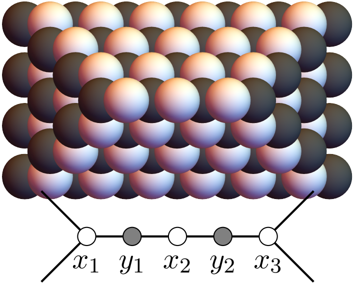
- (optional) final step: truncate by adding negative crystals



Any simply-connected subcrystal can be decomposed into superpositions of positive/negative crystals.

Galakhov-L-Yamazaki '21

Crystal decomposition — infinite chamber for conifold



- positive crystal: starts at $\boxed{1}$ at x_1, x_2, x_3
- negative crystal: starts at $\boxed{2}$ at y_1, y_2

From subcrystal ${}^\sharp\mathcal{C}$ to ground state charge function ${}^\sharp\psi$

Galakhov-L-Yamazaki '21

- charge function of arbitrary crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle\ , \quad \Psi^{(a)}_{\mathrm{K}}(u) \equiv {}^\sharp\psi^{(a)}_0(z) \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{a\Leftarrow b}(u - h(\boxed{b}))$$

- General representations
- contribution from ground state

$$\text{sub-crystal } {}^\sharp\mathcal{C}: \qquad {}^\sharp\psi^{(a)}_0(z) = \frac{\prod_{\beta=1}^{\mathsf{s}_-^{(a)}}(z - z_{-\beta}^{(a)})}{\prod_{\alpha=1}^{\mathsf{s}_+^{(a)}}(z - \mathfrak{p}_\alpha^{(a)})}$$

$$\text{positive crystal staring at } \boxed{a}: \qquad \text{pole } \mathfrak{p}^{(a)} = h(\boxed{a})$$

$$\text{negative crystal staring at } \boxed{a}: \qquad \text{zero } z_-^{(a)} = h(\boxed{a})$$

c.f. canonical crystal $\mathcal{C}_0$: $\quad \psi_0^{(a)}(z) = \left(\frac{1}{z}\right)^{\delta_{a,\mathfrak{o}}}$

Shifted quiver Yangian

Galakhov-L-Yamazaki '21

- mode expansion of original quiver Yangian

$$\psi^{(a)}(z) = \begin{cases} \sum_{j=-1}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1}} & \text{(w/o compact 4-cycle)} \\ \sum_{j=-\infty}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1}} & \text{(w/ compact 4-cycle)} \end{cases}$$

- change of ground state charge function

$$\psi_0^{(a)}(z) = \left(\frac{1}{z}\right)^{\delta_{a,1}} \quad \Longrightarrow \quad {}^\sharp\psi_0^{(a)}(z) = \frac{\prod_{\beta=1}^{\mathtt{s}_-^{(a)}}(z - z_{-\beta}^{(a)})}{\prod_{\alpha=1}^{\mathtt{s}_+^{(a)}}(z - \mathfrak{p}_\alpha^{(a)})}$$

- mode expansion of shifted quiver Yangian

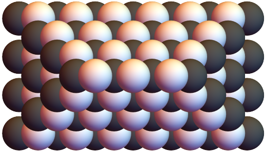
$$\psi^{(a)}(z) = \begin{cases} \sum_{j=-1}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1+\mathtt{s}^{(a)}}} & \text{(w/o compact 4-cycle)} \\ \sum_{j=-\infty}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1+\mathtt{s}^{(a)}}} & \text{(w/ compact 4-cycle)} \end{cases}$$

$$\mathtt{s}^{(a)} \equiv \mathtt{s}_+^{(a)} - \mathtt{s}_-^{(a)}$$

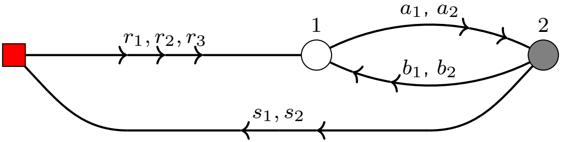
From subcrystal to framed quiver

Galakhov-L-Yamazaki '21

1. For starting atom $\boxed{a}$ of each positive crystal, add an arrow $\infty \to a$
2. For starting atom $\boxed{a}$ of each negative crystal, add an arrow from $a \to \infty$
3. Add terms to superpotential



$Q_3^{(i)} =$



,

$$W_3^{(i)} = \mathrm{Tr}\left[b_2 a_2 b_1 a_1 - b_2 a_1 b_1 a_2 + \sum_{i=1}^{2} s_i \left(a_2 r_i - a_1 r_{i+1}\right)\right].$$

Derivation of shifted quiver Yangians

Galakhov-L-Yamazaki '21

- bootstrapped from action on subcrystals
- confirmed from $\mathcal{N} = 4$ quantum mechanics for framed quiver

Generalize to trigonometric and elliptic version

Galakhov-L-Yamazaki '21

- bond factor

$$\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}} \frac{\prod_{I\in\{a\to b\}} \zeta(u+h_I)}{\prod_{J\in\{b\to a\}} \zeta(u-h_J)}$$

- rational $\longrightarrow$ trigonometric $\longrightarrow$ elliptic

$$\zeta(u) \equiv \begin{cases} u & \text{(rational)} \quad \Longrightarrow \quad \text{quiver Yangians} \\ \sim \sinh \beta u & \text{(trig.)} \quad \Longrightarrow \quad \text{toroidal quiver algebras} \\ \sim \theta_q(u) & \text{(elliptic)} \quad \Longrightarrow \quad \text{elliptic quiver algebras} \end{cases}$$

- Bootstrap from crystal representation before central extension
- Confirm from gauge theory (2D $(2,2)$ and 3D $\mathcal{N}=2$ theory)
- Turn on central extension and fix by consistency

Outline

Summary of construction

Given a toric Calabi-Yau threefold X, consider the BPS sector of D-brane system of type IIA string on X

1. Quiver quantum mechanics (Q, W)

$$\Downarrow \quad \text{define}$$

2. $\{$ BPS states $\} = \{$ colored crystals $\}$

$$\text{act} \quad \Uparrow \Downarrow \quad \text{bootstrap}$$

3. BPS quiver Yangian $Y(Q, W)$

Summary of construction

Given a toric Calabi-Yau threefold X, consider the BPS sector of D-brane system of type IIA string on X

1. Quiver quantum mechanics (Q, W) $\longleftarrow$ Input

   $\Downarrow$ define

2. { BPS states } = { colored crystals }

   act $\Uparrow\Downarrow$ bootstrap

3. BPS quiver Yangian $Y(Q, W)$ $\longleftarrow$ Output

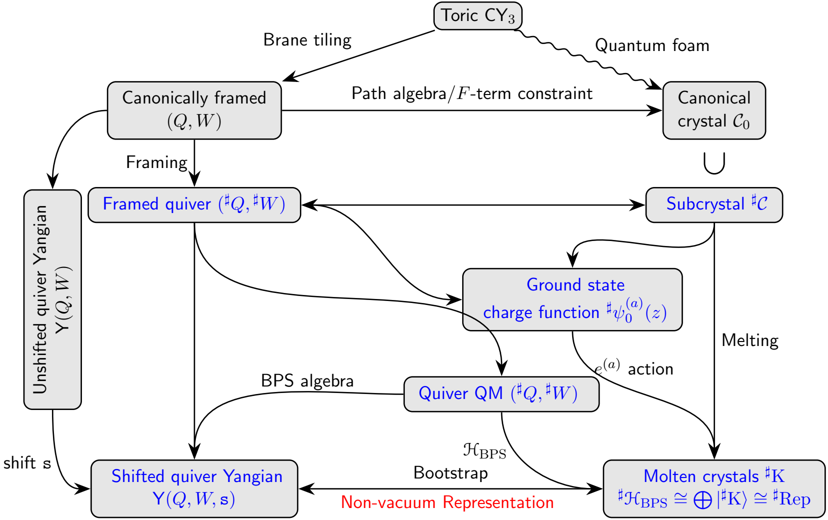
Summary: BPS algebra for general toric Calabi-Yau X_3

1. periodic quiver $(Q, W) \Longrightarrow \varphi^{a\Leftarrow b}(z-w)$ and $|a|$
2. quiver Yangian $Y(Q, W)$

$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,, \\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\,, \\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,e^{(a)}(z)\,, \\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,, \\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,, \\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,,
\end{aligned}
$$

3. Advantages
   - Explicit algebraic relations
   - Apply to ALL toric Calabi-Yau threefolds
4. confirmed from $\mathcal{N}=4$ quiver quantum mechanics

Subcrystal representation, shifted quiver Yangians, and framed quiver



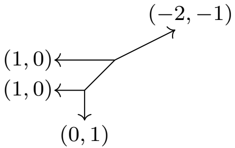
Cover all other cyclic chambers, include open BPS states, and much more

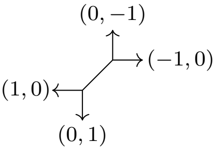
- All have generalization to trigonometric and elliptic version

Future directions

- Relation to gluing constructions

e.g.





L-Longhi '19, L '19

- Translate to $\mathcal{W}$ algebras basis

- Truncation of the algebra

  1. Correspond to non-zero D4 charge
  2. Produce new rational VOA (and their q-deformation and elliptic deformation)
  3. Relations to other systems

More future directions

- meaning of all the new subcrystal representations
- relation to other systems
- generalize to toric Calabi-Yau fourfolds?

Thank you for your attention !

Serre relations

- Demanding that

  vacuum character of algebra = generating function of crystal

  gives additional cubic or higher relations *L-Yamazaki '21*

- Reproduce Serre relations for affine Yangian of $\mathfrak{gl}_{n|m}$

- Open problem: classify Serre relations for general quiver Yangians