Black Hole Interiors as Analytically Continued RG Flow

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(based on arXiv: 2107.00022 and ongoing work
with Elena Cáceres, Arnab Kundu, Ayan K. Patra)

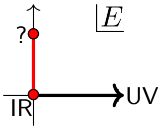
Outline

Section 1

Overview

Analytic Continuation of RG Flows

- In AdS/CFT, data about black hole interiors is encoded in boundary data through analytic continuation.
  - Two-point correlators and the light-cone singularity (AdS_4 or higher) [Fidkowski–Hubeny–Kleban–Shenker 2003]
  - Thermal one-point functions and proper time to the singularity [Grinberg–Maldacena 2020]
- **Proposal:** Consider an RG flow from a UV fixed point to an IR fixed point. Let's complexify the energy after reaching the IR.
  - **"Trans-IR" flow**
  - Natural in picture of holographic RG flow



Holographic RG Flows

- Can think about bulk extra dimension in AdS/CFT as an energy scale for some RG flow from boundary UV theory to IR theory on some radial slice in the bulk [Balasubramanian–Kraus 1999,...]
  - Operator triggering flow dual to bulk field

Accommodating Black Holes

- Radial coordinate of a black hole is spacelike in exterior (UV $\rightarrow$ IR) and timelike in interior (trans-IR)
  - We reach the IR at the *horizon*!
  - **Main takeaway:** Holographic RG flow naturally gives us trans-IR flows because black hole solutions exist.
- **Goal:** Understanding the physics of these trans-IR flows
  - Could teach us more about black hole interiors

Work Done So Far

- *Page Curves and Bath Deformations*, arXiv:2107.00022

  [Cáceres–Kundu–Patra–**SS** 2021]
- Add a Karch-Randall brane to make the holographic RG flow from UV thermal state “doubly holographic”
  - Original motivation: Look at what deforming the bath in the effective picture would do to Page curves/entanglement islands
  - **IR/trans-IR part of the flow pushes the Page time up for faster-running flows.**

Current Objectives (in progress)

- More generally want to understand the trans-IR flow's properties and **how** it encodes the interior
  - We guess and test a candidate "thermal" analog to the holographic monotonic a-function of more typical flows
    [Freedman–Gubser–Pilch–Warner 1999, Myers–Sinha 2010,...]
  - Still seeking better justification and a physical interpretation of our function
    - Covariant bulk objects (null hypersurfaces?)
    - QFT information quantities (complexity, entropy?) [Myers–Sinha 2010,...]

Section 2

Machinery

Scalar Deformations & Radial Ansatz

- In bulk, consider Einstein gravity + (free) scalar

$$I_S = -\frac{1}{4\ell_P^{d-1}} \int d^{d+1}x\sqrt{-g}\left(\nabla^\alpha\phi\nabla_\alpha\phi + m^2\phi^2\right) \tag{2.1}$$

- $m^2 < 0$ scalar field is dual to a relevant operator $\mathcal{O}$ on the boundary:

$$m^2 = \Delta(\Delta - d) \tag{2.2}$$

- Consider black hole solutions to Einstein + scalar with "Schwarzschild-like" ansatz and radial field $\{F, \chi, \phi\}$:

$$ds^2 = \frac{1}{r}\left[-F(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{F(r)} + d\vec{x}^2\right], \quad \phi = \phi(r) \tag{2.3}$$

Numerical Solutions

- Equations of motion for radial ansatz:

$$\phi'' + \left(\frac{F'}{F} - \frac{d-1}{r} - \frac{\chi'}{2}\right)\phi' + \frac{\Delta(d-\Delta)}{r^2 F}\phi = 0 \tag{2.4}$$

$$\chi' - \frac{2F'}{F} - \frac{\Delta(d-\Delta)\phi^2}{(d-1)rF} - \frac{2d}{rF} + \frac{2d}{r} = 0 \tag{2.5}$$

$$\chi' - \frac{r}{d-1}(\phi')^2 = 0 \tag{2.6}$$

- Perform a numerical two-sided shooting method from the horizon
  - Each flow labeled by dimensionless "deformation parameter" $\phi_0/T^{d-\Delta}$

Kasner Universes

- Reparameterizing r, near-singularity metric looks like:

$$ds^2 \sim -d\tau^2 + \tau^{2p_t} dt^2 + \tau^{2p_x} d\vec{x}^2, \quad \phi \sim \sqrt{2} p_\phi \log \tau \tag{2.7}$$

  - $\{p_t, p_x, p_\phi\}$ classically satisfy:

$$p_t + (d-1)p_x = 1, \quad p_t^2 + (d-1)p_x^2 + p_\phi^2 = 1 \tag{2.8}$$

    - These are thus **Kasner exponents**
    - Get a **Kasner universe** near the singularity
      $\longrightarrow$ "Kasner flows" [Frenkel–Hartnoll–Kruthoff–Shi 2020]

Section 3

Example: Page Curves and Page Times

Karch-Randall Branes

- Add an end-of-the-world (EOW) **Randall-Sundrum (RS) brane** $\mathcal{Q}$ with tension T_{RS} to the theory, [Randall–Sundrum 1999]

$$I_{RS} \sim \int_{\mathcal{Q}} d^d x \sqrt{-h}(K - T_{RS}) \implies K_{ab} = (K - T_{RS}) h_{ab} \tag{3.1}$$

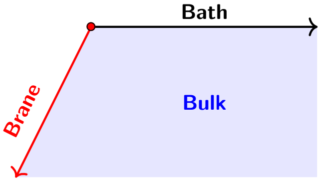
- Take tension to be subcritical; in units of $16\pi G_N$,

$$|T_{RS}| < d - 1 \tag{3.2}$$

  - This is a **Karch-Randall (KR) brane** [Karch–Randall 2001]
  - Induced geometry is asymptotically AdS$_d$

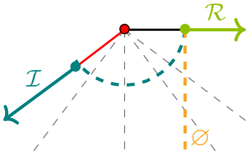
Double Holography

- Three equivalent pictures [Karch–Randall 2001] [Takayanagi 2011]:
  - **Bulk:** $(d+1)$-dimensional Einstein gravity with EOW KR brane
  - **Brane + Bath:** CFT$_d$ coupled to effective gravity on asymptotically AdS$_d$ space (brane), with a transparent interface connecting to non-gravitating CFT$_d$ "bath" $\rightarrow$ **Page curves here**
  - **QFT:** d-dimensional **boundary** CFT (BCFT$_d$)



Island Rule

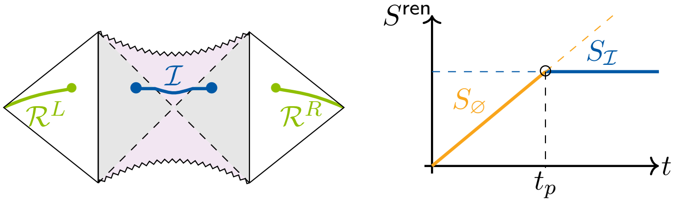
- Double holography is nice for studying islands in $d > 2$ brane + bath [Almheiri–Mahajan–Santos 2019, Geng–Karch 2020, Geng–Karch–Pérez-Pardavila–Raju–Randall–Riojas–**SS** 2020-21,...]
  - **Quantum extremal surfaces** [Engelhardt–Wall 2014] of brane + bath are encoded by **classical extremal surfaces** in the bulk
  - **Island rule:** For "radiation" interval $\mathcal{R}$ in bath, entropy is computed in bulk as the minimal area over all extremal surfaces γ homologous to $\mathcal{R}$ and *any* subinterval $\mathcal{I}$ of brane **("minimization chooses the island")**



$$S(\mathcal{R}) = \min_{\mathcal{I}} \text{ext} \frac{A(\gamma)}{4G_N}, \quad \gamma \sim \mathcal{R} \cup \mathcal{I} \tag{3.3}$$

Computing "Eternal" Page Curves

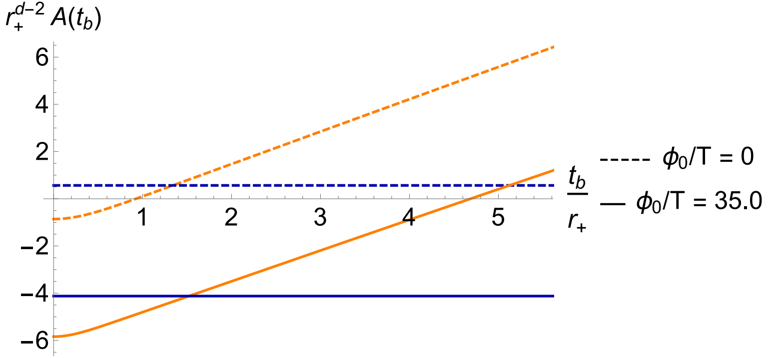
- Consider eternal black holes and radiation $\mathcal{R} = \mathcal{R}^L \cup \mathcal{R}^R$ collected on both sides [Almheiri–Mahajan–Maldacena 2019]



- Phase transition of bulk's classical entanglement surfaces (corresponding to emergence of an island)
  - Should and will be informed by classical backreaction

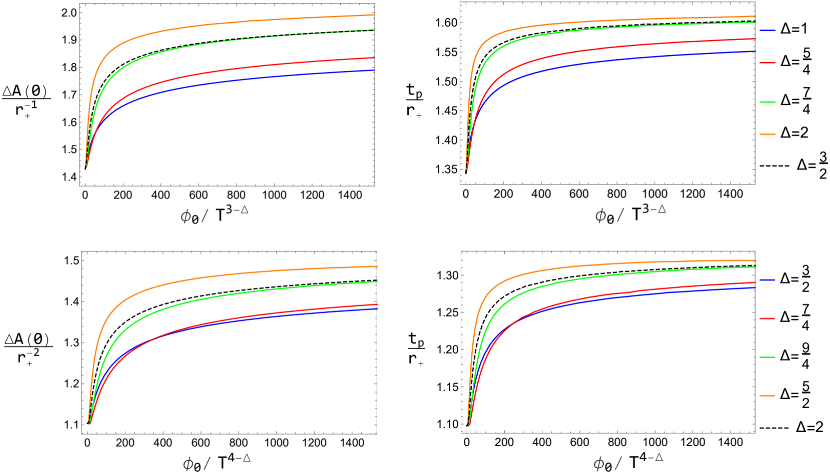
Computing Page Curves (cont.)

- **Example:** Different scalar deformations with $d = 3$, $\Delta = 2$, fixed $\mathcal{R}$



Plots on Plots

- Two features of Page curve: initial area difference and Page time



RG Flow Interpretation

- Fix d, Δ: Each choice of $\phi_0/T^{d-\Delta}$ determines a particular flow, including trans-IR sector behind the horizon
- Also fix $\mathcal{R}$: Page curve **reflects which (trans-IR) flow we are on**!
  - Consequence of 1-1 behavior seen in numerics

Increasing $\phi_0/T^{d-\Delta}$ $\implies$ Stronger effects from the IR/trans-IR regimes relative to UV (i.e. "faster coarse-graining")

  - Like BCFT result of [Roazli–Sully–Van Raamsdonk–Waddell–Wakeham 2019]:

$$t_p \sim \frac{c_{d-1}}{c_d}. \tag{3.4}$$

Section 4

The Monotonic a_T-Function (work in progress)

"Blackened" Domain Wall Ansatz

- Consider the $(d+1)$-dimensional "blackened" domain wall ($L_{\text{AdS}} = 1$)

$$ds^2 = e^{2A(\rho)} \left[-f(\rho)^2 dt^2 + d\vec{x}^2\right] + d\rho^2 \tag{4.1}$$

  - Setting $f(\rho) = 1$ reproduces a domain wall with flat slicing in which holographic a-function is known as [Freedman–Gubser–Pilch–Warner 1999]

$$a(\rho) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right) \ell_P^{d-1}} \left[\frac{1}{A'(\rho)}\right]^{d-1} \implies a_{UV} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right) \ell_P^{d-1}} \tag{4.2}$$

  - Null energy condition (NEC) implies monotonicity in **any** d

"Blackened" Domain Wall Ansatz (cont.)

- We care about black holes (i.e. f has a simple root at $\rho = 0$), so we propose a simple **guess** for a modification.

$$\boxed{a_T(\rho) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}}\left[\frac{f(\rho)}{A'(\rho)}\right]^{d-1}} \tag{4.3}$$

  - Still thinking about other rationale
  - **Sanity check:** This is stationary for AdS-Schwarzschild (functions presented in [Hartman–Maldacena 2013]).

$$e^{A(\rho)} = \frac{2}{d}\cosh\left(\frac{d\rho}{2}\right)^{2/d}, \quad f(\rho) = \tanh\left(\frac{d\rho}{2}\right). \tag{4.4}$$

  - Reproduces same value as empty AdS for previous a-function

a_T in Schwarzschild-like Coordinates

- Write a_T-function in terms of r:

$$\boxed{a_T(r) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}} e^{-(d-1)\chi(r)/2}} \tag{4.5}$$

- The energy scale is still ρ! Need to use chain rule to write:

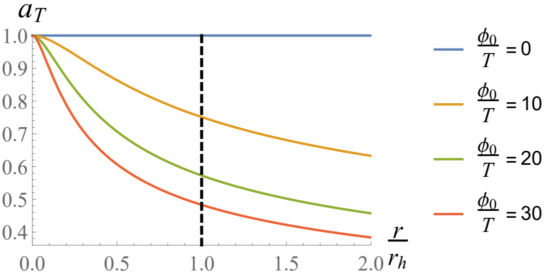
$$\frac{da_T}{d\rho} = \frac{dr}{d\rho}\frac{da_T}{dr} = -r\sqrt{F(r)}\frac{da_T}{dr} \tag{4.6}$$

- Can prove monotonicity **even along the trans-IR** ($\rho = i\kappa$, $\kappa > 0$)!

$$\boxed{\text{Exterior:} \quad \frac{da_T}{d\rho} \geq 0, \quad \text{Interior:} \quad \frac{da_T}{d\kappa} \leq 0} \tag{4.7}$$

Plotting $a_T(r)$ for Kasner Flows

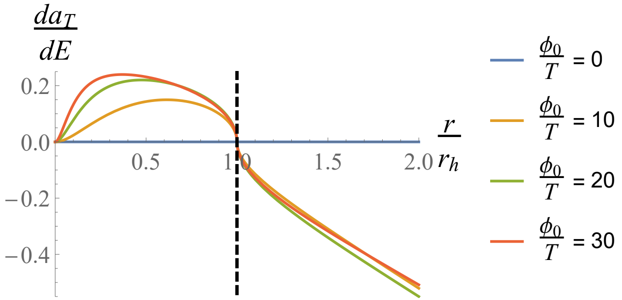
- From the $d = 3$, $\Delta = 2$ shooting, obtain $a_T(r)$ (in units of $2\pi\ell_P^{-2}$ for a family of flows



- Horizon is dashed line; UV $\rightarrow$ IR regime is to the left while trans-IR (IR $\rightarrow$ singularity) is to the right

Plotting Derivative of a_T

- Can also plot the derivative of a_T along the flow



- Monotonicity is indeed obeyed!

Section 5

Conclusions

The Story Thus Far...

- Trans-IR flows are natural in holographic RG flow as black hole interiors
  - Have examined how Page time is a "probe" of such flows
  - We have proposed a simple monotonic a_T-function (in Einstein gravity) which also works behind the horizon
  - Have used known Kasner flows as a test case

Next Steps for Us

- "How can we further justify our a_T-function?" (Some covariant quantity/null hypersurfaces?)
- "What does our a_T-function physically represent?" (Entanglement? Complexity?)
- "What are we actually doing to the state?" (Some sort of Euclidean evolution?)

Thank you!

Relevant Deformations of the UV

- AdS/CFT dictionary: Fields in the bulk are dual to operators on the boundary
- Add a field dual to a *relevant* operator
    - Field should be normalizable
    - Triggers a holographic RG flow from the UV theory/state

Labeling Kasner Flows

- Each Kasner flow is labaled by a dimensionless **deformation parameter** $\phi_0/T^{d-\Delta}$
  - T is the black hole temperature:

$$T = \frac{|f'(r_h)|e^{-\chi(r_h)/2}}{4\pi} \tag{5.1}$$

  - ϕ_0 is the source read from the near-boundary expansion of the scalar field

$$\phi(r) \sim \phi_0 r^{d-\Delta} + \frac{\langle \mathcal{O} \rangle}{2\Delta - d} r^{\Delta}, \quad r \to 0 \tag{5.2}$$

  - In practice, get this numerically for a given shooting problem with a particular numerical value of the horizon

Deforming the Bath

- **Question:** Does the bath affect the physics?
  - Gives us **massive** gravity in the braneworld, which might throw a wrench into the higher-dimensional island story [Geng and Karch, 2020] [Geng, Karch, Pérez-Pardavila, Raju, Randall, Riojas, **SS**, 2020-21]
- Another approach: **deforming the bath**
  - We did a preliminary study using a **scalar** deformation $(T_{RS} = 0)$ [Cáceres, Kundu, Patra, **SS**, 2021]
  - Backreaction is classical $\implies$ QES in brane + bath should be affected
  - **The full holographic RG flow (including trans-IR) will inform the Page curve**

Brief History of Monotonicity

- **Zamolodchikov's 2D c-theorem (1986):** Can define a c-function which monotonically decreases as we RG flow from a UV CFT fixed point to an IR CFT fixed point

$$[c]_{\mathsf{UV}} \geq [c]_{\mathsf{IR}} \tag{5.3}$$

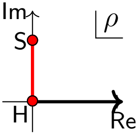
  - Only needs unitarity, Lorentz invariance, and renormalizability
  - c coincides with **central charge** at endpoints
- **Cardy's even-dimension a-theorem (1988):** Extended Zamolodchikov's result to RG flows of even-dimensional QFT
  - Defined an a-function coinciding with a-central charge at endpoints arising in trace anomalies
  - Was later done holographically [Freedman, Gubser, Pilch, Warner, 1999]

Brief History of Monotonicity (cont.)

- **Myers and Sinha's holograhic a-theorem (2010):** Took a-function of [Freedman, Gubser, Pilch, Warner, 1999] **on gravity side** (in $d+1$ dimensions)
  - Identified as a central charge appearing in the coefficient of the universal term of entanglement entropy (e.g. $(c/6)\log(l/\epsilon)$)
  - No trace anomaly for odd d, but the entanglement entropy view justifies this as an "odd-d central charge"
- **This is the function we will care about.**

Problems with the Blackened Domain Wall

- We access interior by analytical continuation of coordinates ($\kappa > 0$):



$$\boxed{\rho = i\kappa}, \quad t = t_I - \frac{i\pi}{2} \tag{5.4}$$

- Ultimately, we would like to prove the following:

$$\text{Exterior:} \quad \frac{da_T}{d\rho} \geq 0 \tag{5.5}$$

$$\text{Interior:} \quad \frac{da_T}{d\kappa} \leq 0 \tag{5.6}$$

  - The interior inequality is tricky in blackened domain wall

Switching to Schwarzschild-like

- Put interior AND exterior on a single real coordinate

$$e^{2A(\rho)} = \frac{1}{r^2}, \quad f(\rho)^2 = e^{-\chi(r)} F(r), \quad \frac{dr}{d\rho} = -r\sqrt{F(r)} \tag{5.7}$$

  - $r \geq 0$ (with $r = 0$ the boundary); $F(r)$ has a simple root at $r = r_h$

- Resulting metric is Schwarzschild-like

$$ds^2 = \frac{1}{r^2}\left[-F(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{F(r)} + d\vec{x}^2\right] \tag{5.8}$$

a_T in Schwarzschild-Like Coordinates

- Can write derivative of a_T in terms of stress tensor

$$\frac{da_T}{dr} = -\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\frac{e^{-(d-1)\chi(r)/2}}{rF(r)^2}\left[F(r)\left(T^r{}_r - T^t{}_t\right)\right] \tag{5.9}$$

- The NEC implies that

$$\vec{k} = e^{\chi(r)/2}\partial_t + F(r)\partial_r \implies F(r)\left(T^r{}_r - T^t{}_t\right) \geq 0 \tag{5.10}$$

$$\implies \frac{da_T}{dr} \leq 0 \tag{5.11}$$

Monotonicity of a_T

- Proof in **exterior** is straightforward from NEC:

$$\frac{dr}{d\rho} \leq 0 \implies \frac{dr}{d\rho}\frac{da_T}{dr} \geq 0 \tag{5.12}$$

- Proof in **interior** is trickier because we need $dr/d\kappa$; as $F(r) < 0$:

$$\begin{aligned} \frac{1}{i}\frac{dr}{d\kappa} &= -r\sqrt{F(r)} = -ir\sqrt{|F(r)|} \\ \implies \frac{dr}{d\kappa} &= r\sqrt{|F(r)|} \geq 0 \\ \implies \frac{dr}{d\kappa}\frac{da_T}{dr} &\leq 0 \end{aligned} \tag{5.13}$$

Horizon is the IR Fixed Point

- **Sanity check:** a_T should be stationary at IR fixed point
- First write

$$\frac{da_T}{dr} \propto \chi'(r)e^{-(d-1)\chi(r)/2} \tag{5.14}$$

  - Note that χ is at horizon, so this is finite
- At the horizon, $dr/d\rho = 0$ because $F(r_h) = 0$
  - Immediately have that $da_T/d\rho = 0$ at horizon!

a_T at the Kasner Singularity

- Can analytically get the near-singularity behavior of a_T in terms of the parameter,

$$c^2 = \frac{2[d(1+p_t)-2]}{(d-1)(1-p_t)} \geq 0 \tag{5.15}$$

- Up to an overall positive prefactor,

$$a_T(r) \sim r^{-(d-1)^2 c^2/2}, \quad r \to \infty \tag{5.16}$$

  - For the nontrivial flows, a_T goes to 0 at the singularity.

a_T Near the Kasner Singularity

- Again up to an overall positive prefactor,

$$\frac{da_T}{d\kappa} \sim -r^{\sigma}, \quad r \to \infty, \tag{5.17}$$

$$\sigma = \frac{(d-1)[d(1+p_t)-3]}{p_t - 1} \tag{5.18}$$

- Currently unclear if there is an analytic argument for the sign of σ; numerics hint it to be generically positive
  - Analytically know that $p_t > -1 + 2/d$ (Schwarzschild value)
  - *Numerics* $(d = 2, 3, 4)$ indicate $p_t < -1 + 3/d$ [Frenkel, Hartnoll, Kruthoff, Shi, 2020] [Cáceres, Kundu, Patra, **SS**, 2021]