A KRYLOV SPACE ODYSSEY

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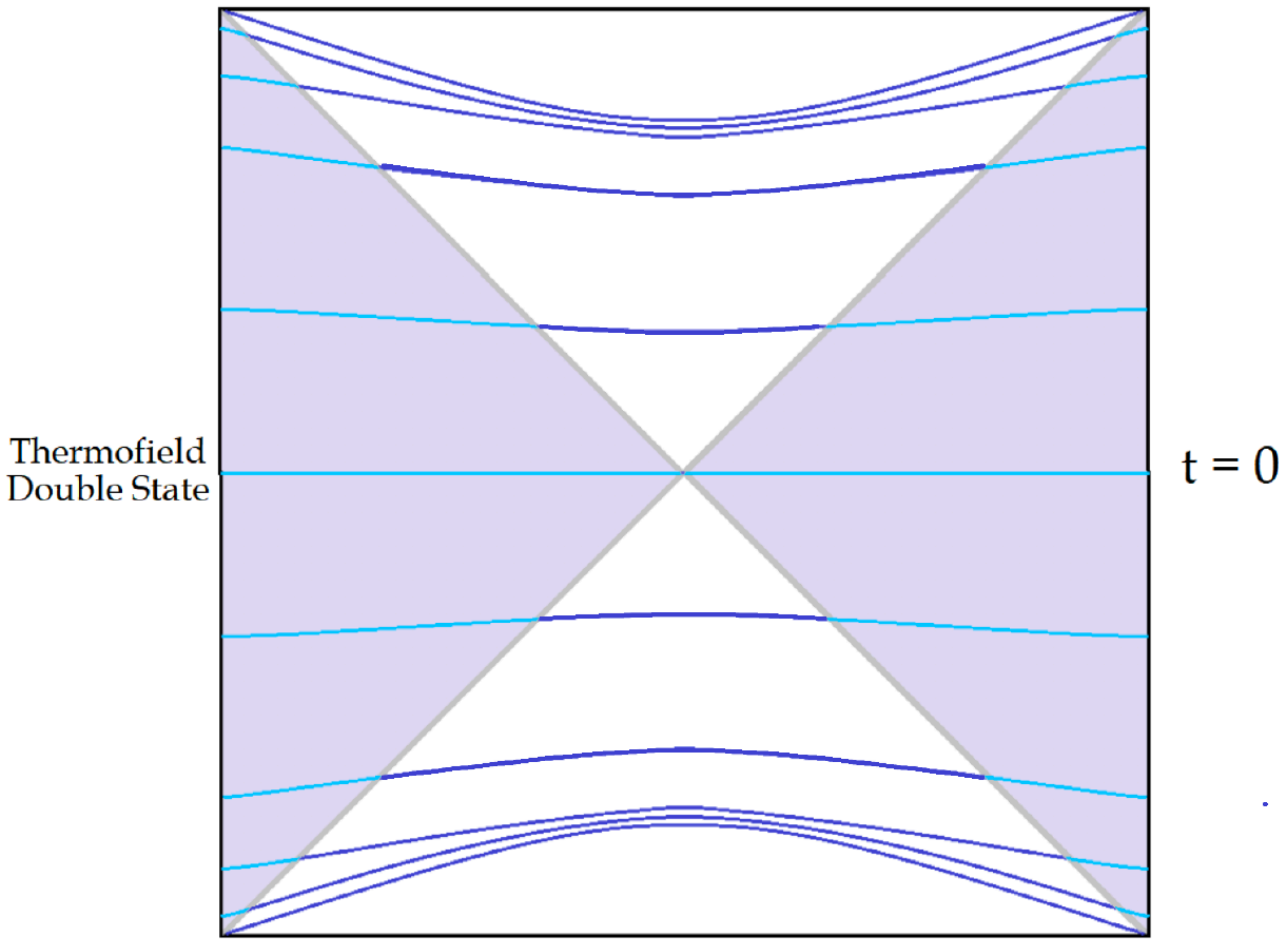
COMPLEXITY IN HOLOGRAPHY?



(L. Susskind, 2018.)

- What boundary observable describes the growth of the Einstein-Rosen Bridge (ERB)?
- It *could* be complexity *(L. Susskind, 2013)*.

But what complexity??

➤ Should saturate at Heisenberg time, $t_H \sim e^S$.

➤ No *external* tolerance parameter.

STATES OR OPERATORS?

- ERB of eternal AdS black hole should be captured by the evolution of $|TFD\rangle$ in the **Schrödinger picture**.
- *Butterfly effect:* backreacted geometry due to shockwave of infalling particle *(Shenker & Stanford, 2013)*.

$\longrightarrow$ Captured by evolution of $|TFD\rangle$ perturbed by an operator insertion $\mathcal{O}(t)$.

$\longrightarrow$ Motivates the study of **operator complexity in the Heisenberg picture**.

This gives access to a wide range of time scales:

➤ **Scrambling time,** $t_s \sim \log S$ *(infalling particle crosses the BH horizon)*.

➤ **Heisenberg time,** $t_H \sim e^S$ *(geometrical description fails, Hilbert space fully explored, fine-grained spectrum is probed…)*

KRYLOV SPACE

- A notion of complexity adapted to time evolution of an initial operator $\mathcal{O} \equiv \mathcal{O}(0)$.

- Take a Hilbert space of states $\mathcal{H}$ with $\dim \mathcal{H} = D$.

$\implies$ Operator space $\widehat{\mathcal{H}}$ will have $\dim \widehat{\mathcal{H}} = D^2$.

- Time evolution generator in $\widehat{\mathcal{H}}$ is the **Liouvillian** $\mathcal{L} := [H, \cdot\,]$, as:

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt} = e^{it\mathcal{L}}\mathcal{O} = \mathcal{O} + it[H, \mathcal{O}] - \frac{t^2}{2}\big[H, [H, \mathcal{O}]\big] + \dots$$

- Define **Krylov space** as $\mathcal{H}_{\mathcal{O}} := \text{span}\left\{\mathcal{L}^n\mathcal{O}\right\}_{n=0}^{+\infty}$

$\implies$ Always contains $\mathcal{O}(t)$. What is its dimension, K??

KRYLOV SPACE DIMENSION AND UPPER BOUND

The spectrum of $\mathscr{L}$ is made up of all possible energy differences $E_a - E_b \equiv \omega_{ab}$:

$$|E_a\rangle\langle E_b| \equiv |\omega_{ab}) \implies \mathscr{L}|\omega_{ab}) = \omega_{ab}|\omega_{ab})$$

Spectral decomposition of $|\mathcal{O}) = \sum_{ab} O_{ab}|\omega_{ab}) \implies \mathscr{L}^n|\mathcal{O}) = \sum_{ab} \omega_{ab}^n O_{ab}|\omega_{ab})$

$\implies K =$ Number of non-zero eigenspace projections of $|\mathcal{O})$!!

Immediate upper bound: $K \leq D^2 - D + 1$

- Saturated in chaotic systems (ETH).
- Degeneracies reduce K

LANCZOS COEFFICIENTS AND KRYLOV CHAIN DYNAMICS

- We can build and *ordered* orthonormal basis for Krylov space **(Lanczos algorithm)**.

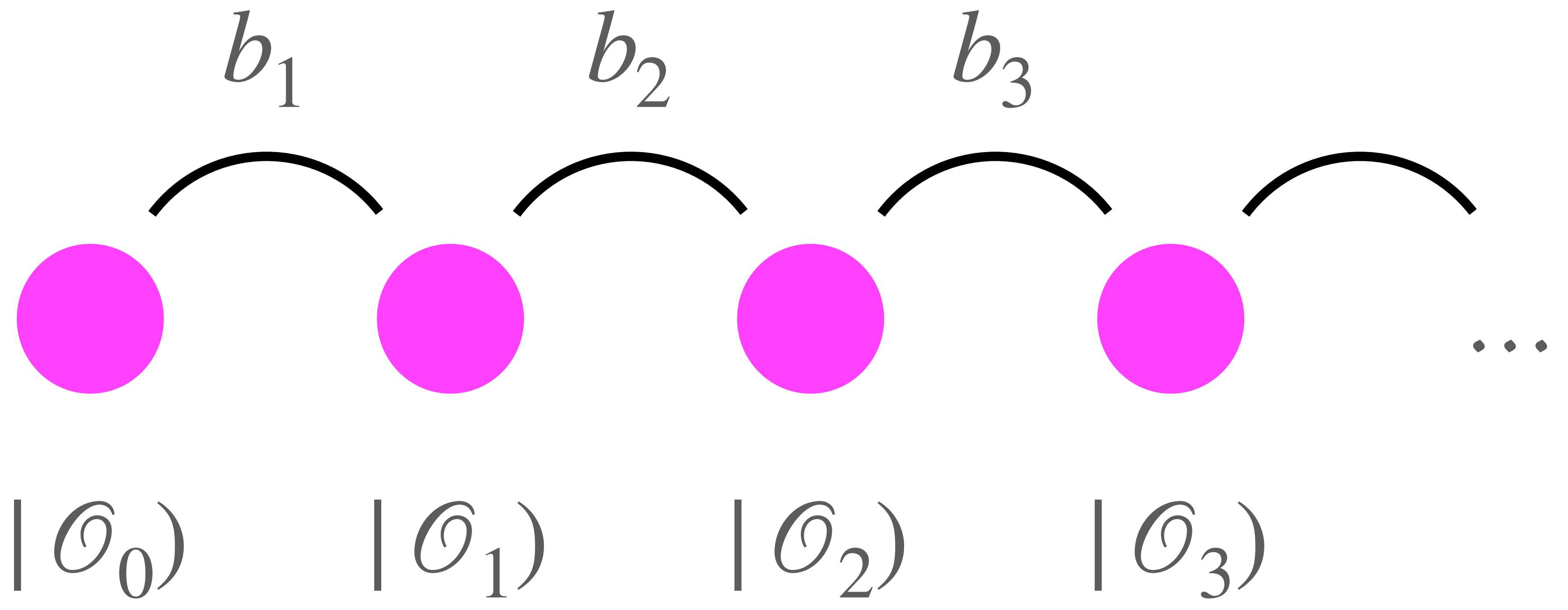
$$\{ |\mathcal{O}), \mathcal{L}|\mathcal{O}), \mathcal{L}^2|\mathcal{O}), \dots \} \longmapsto \{ |\mathcal{O}_0) = |\mathcal{O}), |\mathcal{O}_1), |\mathcal{O}_2), \dots \}$$

Krylov elements

- Re-orthogonalization coefficients: **Lanczos coefficients**, $\{b_n\}$.

$\Longrightarrow$ Liouvillian is tri-diagonal!

$$(\mathcal{L}_{mn}) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \dots & 0 \\ b_1 & 0 & b_2 & 0 & \dots & 0 \\ 0 & b_2 & 0 & b_3 & \dots & 0 \\ \vdots & & \ddots & & \ddots & \vdots \end{pmatrix}$$



1d hopping model!

K-COMPLEXITY AND K-ENTROPY

- Evolution of $|\mathcal{O}(t))$ encoded in wave function $\varphi(t)$:

$$|\mathcal{O}(t)) = \sum_n i^n \varphi_n(t) |\mathcal{O}_n)$$

K-COMPLEXITY:

$$C_K(t) = \sum_n n \left|\varphi_n(t)\right|^2$$

K-ENTROPY:

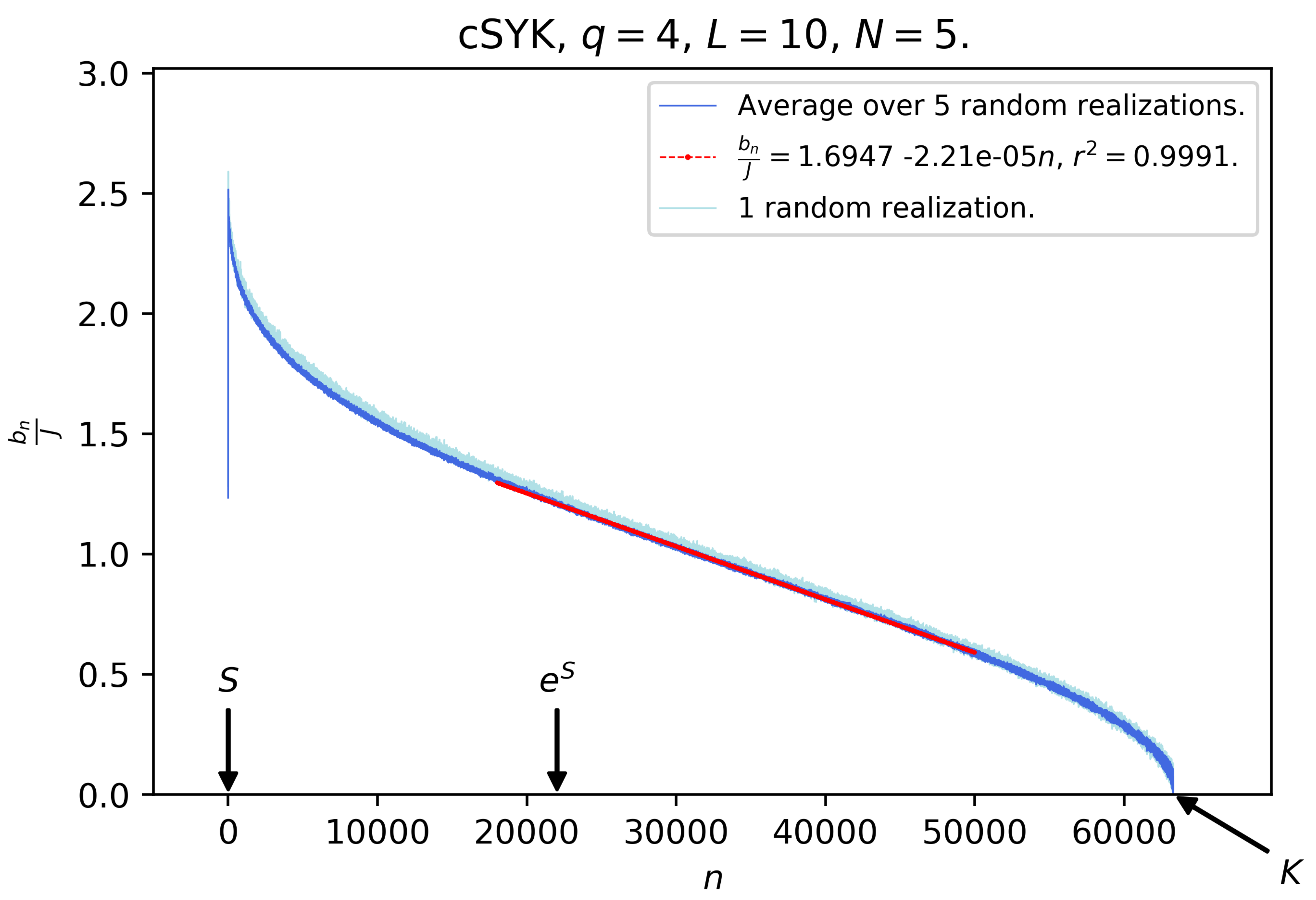
$$S_K(t) = -\sum_n \left|\varphi_n(t)\right|^2 \log \left|\varphi_n(t)\right|^2$$

For **maximally chaotic systems:**

-Thermodynamic limit *(Altman et al., 2019)*: $b_n \sim \alpha n, \quad C_K(t) \sim e^{2\alpha t}, \; S_K(t) \sim t$

-Finite size, post-scrambling *(Barbón et al., 2019)*: $b_n \to b_\infty, \; C_K(t) \sim t, \; S_K(t) \sim \log(t)$

LANCZOS COEFFICIENTS IN SYK

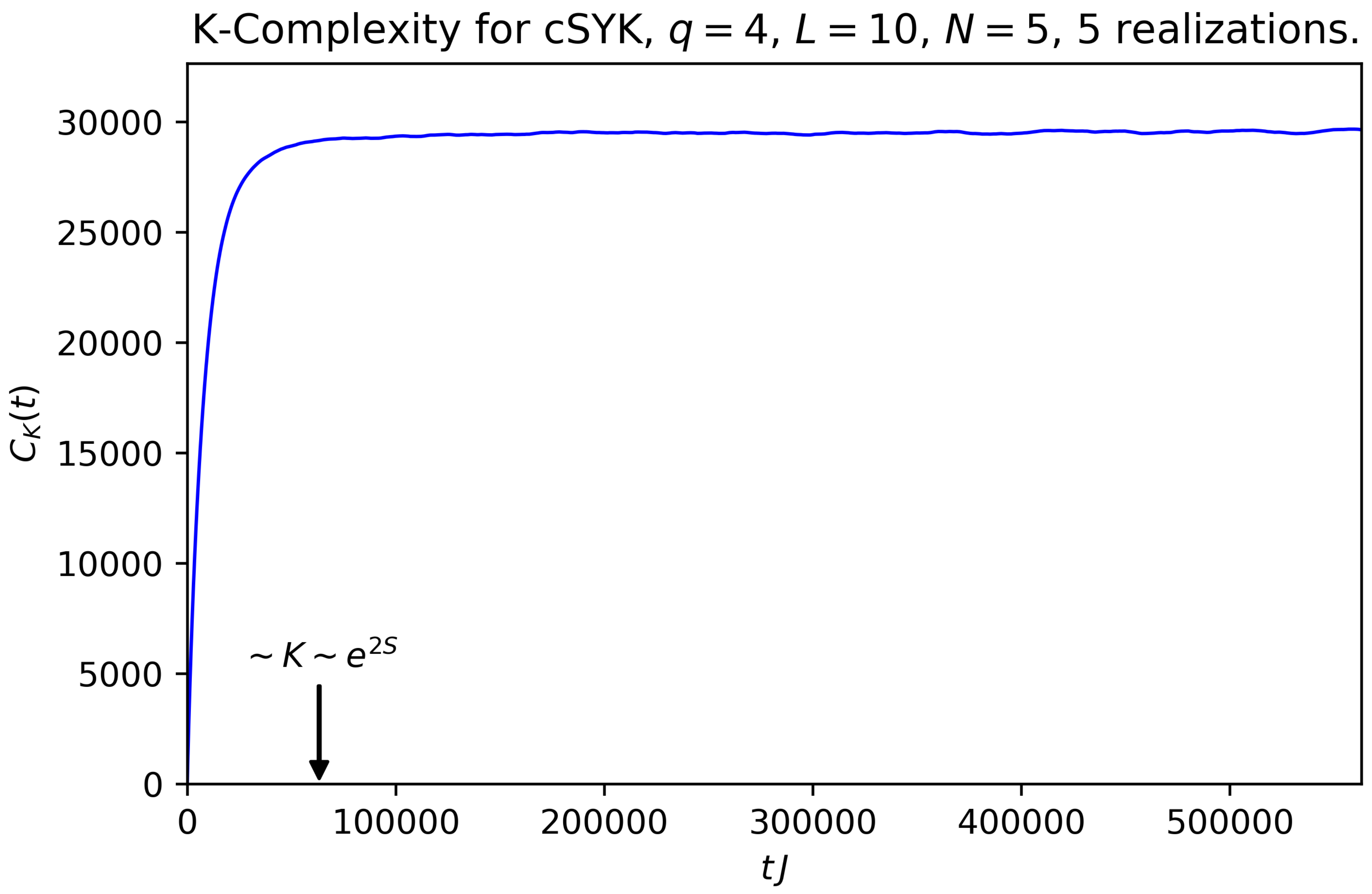


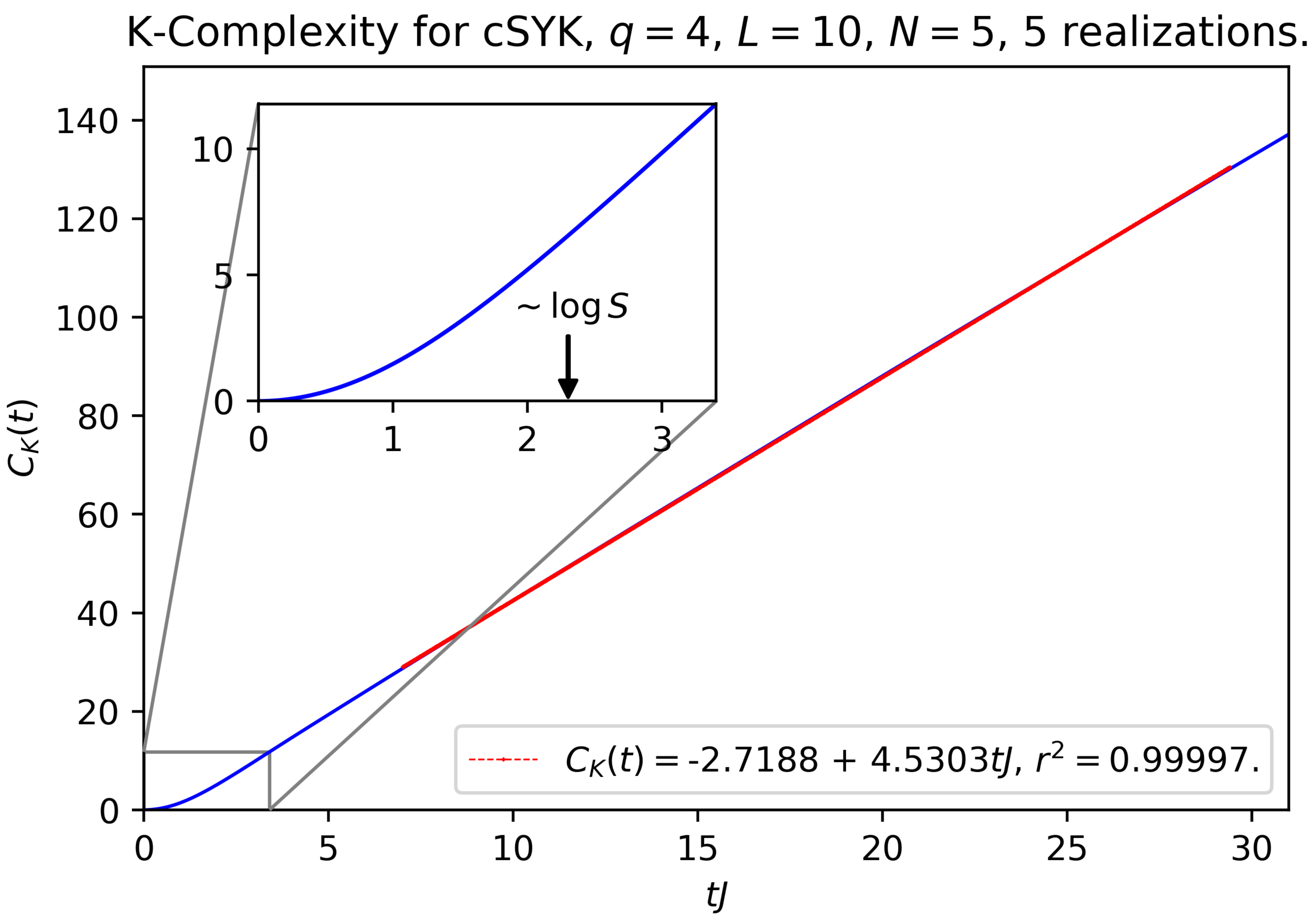
$$H = \sum_{ijkl} J_{ij;kl}\, c_i^\dagger c_j^\dagger c_k c_l \,; \quad \mathcal{O} = c_1^\dagger c_2 + \text{h.c.}$$

- Upper bound saturated, $K = D^2 - D + 1$.
- Linear growth up to $n \sim S$.
- **Non-perturbative descent** with slope $\sim e^{-2S}$.
- Reaches zero at $n = K \sim e^{2S}$.

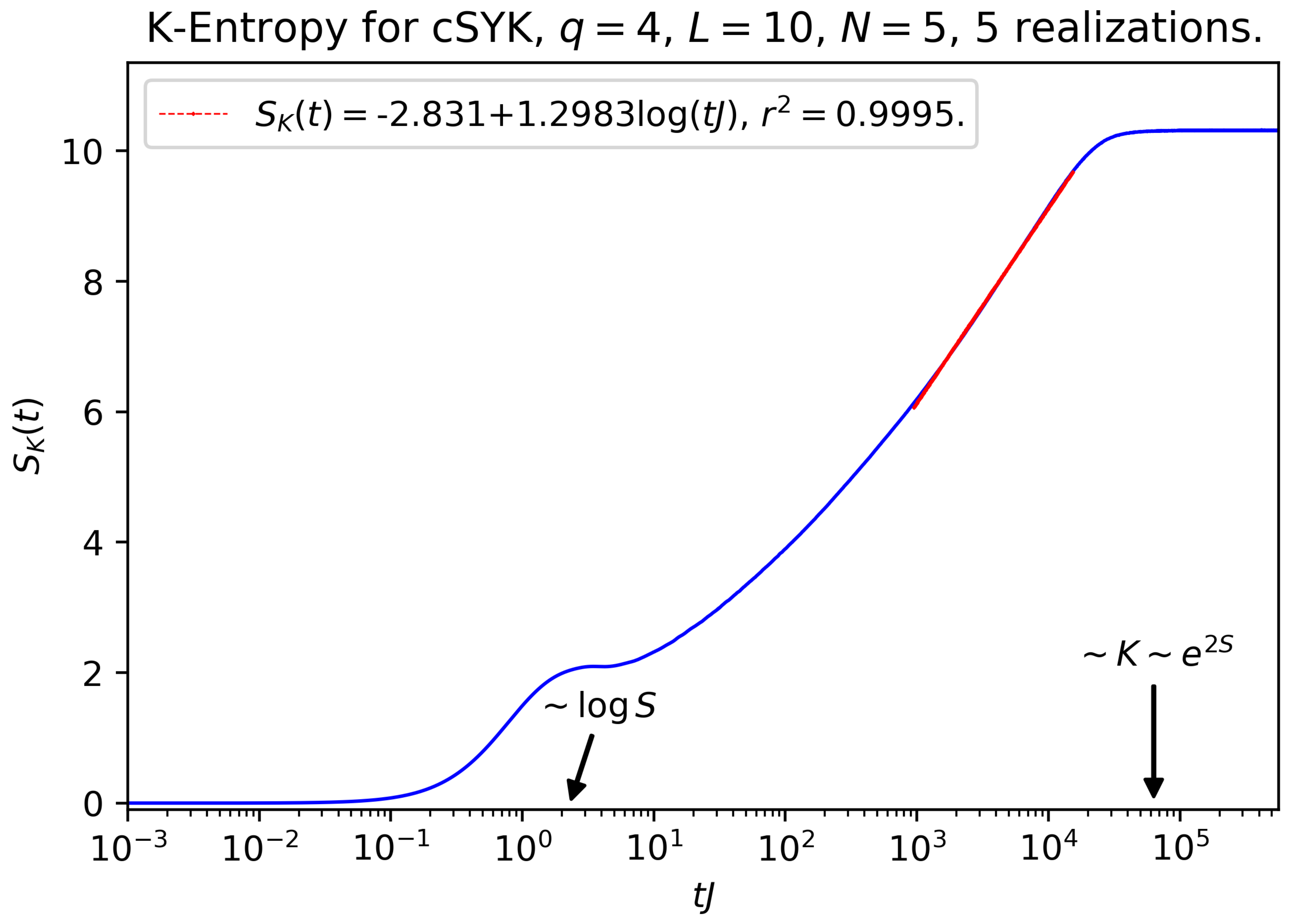
(E. Rabinovici, ASG, R. Shir, J. Sonner, 2020.)

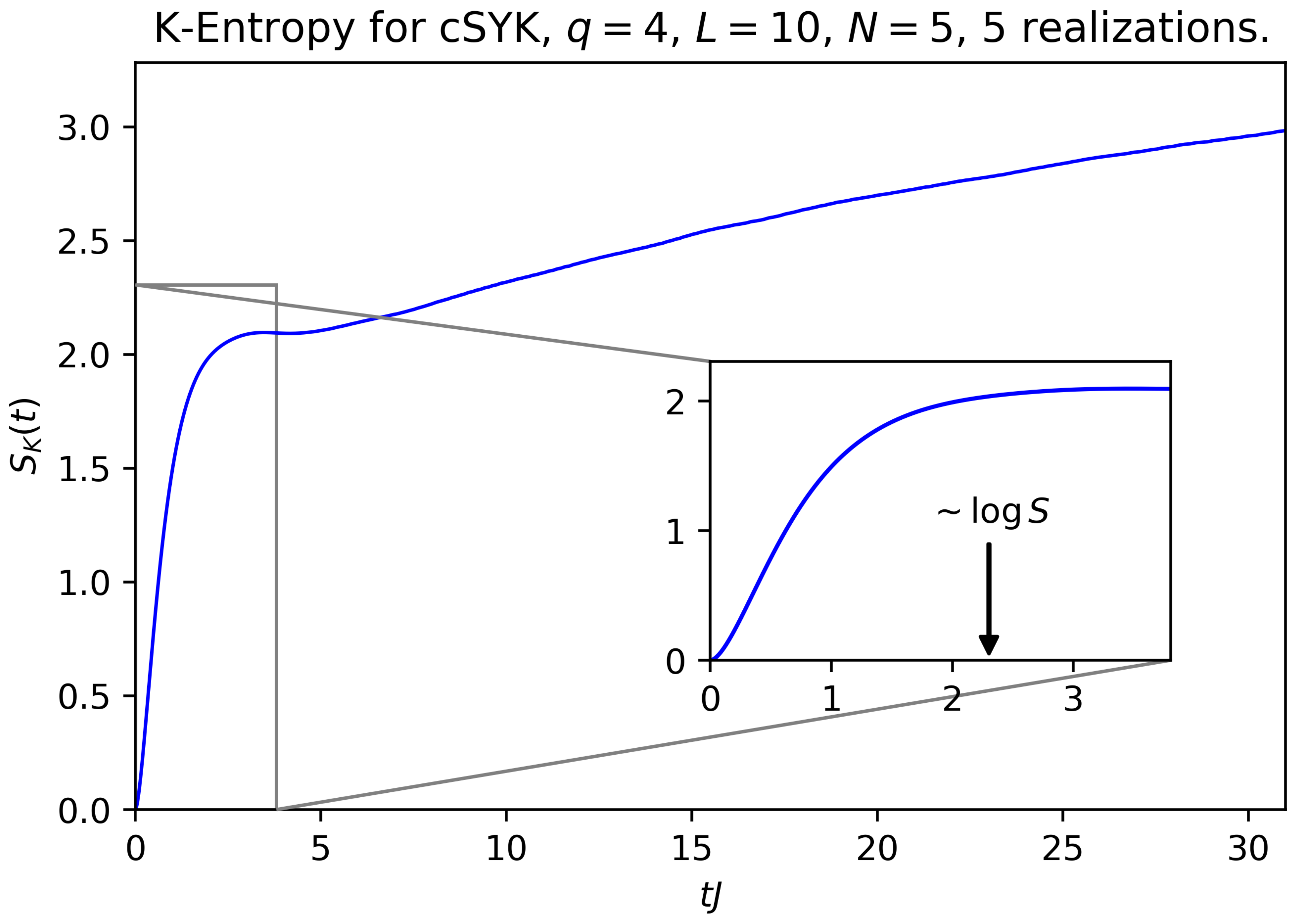
K-COMPLEXITY IN SYK





K-ENTROPY IN SYK





CONCLUSIONS

- K-complexity is a notion of complexity that doesn't depend on an external tolerance parameter.
- It is very sensitive to the spectral structure.
- Non-perturbative descent of $b_n \longrightarrow$ Boundedness and saturation of $C_K(t)$.
- Some questions:
  - Bulk interpretation?
  - Problems: Initial condition dependence; time-dependent Hamiltonians…
  - Probe of quantum chaos? *(Dymarsky & Smolkin, 2021; work in progress…)*

Late time physics!

¡ Thanks !

KRYLOV SPACE DIMENSION – VANDERMONDE ARGUMENT

- Liouvillian eigenvalues: $\omega_{ab} = E_a - E_b \implies \omega_{aa} = 0, \ \forall a = 1, \ldots, D.$

$$\implies \mathscr{L}^n |\mathcal{O}) = \delta_{n0} \sum_{a=1}^{D} O_{aa} |\omega_{aa}) + \sum_{a \neq b} \omega_{ab}^n O_{ab} |\omega_{ab})$$

- In coordinates:

$$\begin{pmatrix} \mathcal{O} \\ \mathscr{L}\mathcal{O} \\ \mathscr{L}^2\mathcal{O} \\ \vdots \end{pmatrix} \overset{*}{=} \begin{pmatrix} O_{11} & O_{22} & \ldots & O_{DD} & O_{12} & O_{13} & \ldots & O_{D-1,D} \\ 0 & 0 & \ldots & 0 & O_{12}\,\omega_{12} & O_{13}\,\omega_{13} & \ldots & O_{D-1,D}\,\omega_{D-1,D} \\ 0 & 0 & \ldots & 0 & O_{12}\,\omega_{12}^2 & O_{13}\,\omega_{13}^2 & \ldots & O_{D-1,D}\,\omega_{D-1,D}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

$\implies$ Vandermonde matrix!!

LANCZOS ALGORITHM

Construct orthonormal basis **given some inner product,** e.g. $\left(A|B\right) = \frac{1}{D}\mathrm{Tr}\left[A^{\dagger}B\right]$.

1. Set $b_0 \equiv 0$ and $|\mathcal{O}_{-1}) \equiv 0$.
2. $|\mathcal{O}_0) = \frac{1}{\sqrt{(\mathcal{O}|\mathcal{O})}}|\mathcal{O})$
3. For $n \geq 1$: $|\mathcal{A}_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$
4. $b_n = \sqrt{(\mathcal{A}_n|\mathcal{A}_n)}$
5. If $b_n = 0$ **stop**. Otherwise $|\mathcal{O}_n) = \frac{1}{b_n}|\mathcal{A}_n)$

$\longrightarrow \left\{b_n\right\}_{n=1}^{K-1}$ (Lanczos coefficients)

$\longrightarrow \left\{|\mathcal{O}_n)\right\}_{n=0}^{K-1}$ (Krylov basis)

-Due to linear independence:

$|\mathcal{A}_K) = 0 \implies b_K = 0,$

And the algorithm terminates.

Problem: numerically unstable!!

RELATION BETWEEN LANCZOS COEFFICIENTS AND CORRELATION FUNCTION

Inner product $\longrightarrow$ 2-point function: $\left(A\,|\,B\right) = \frac{1}{D}\mathrm{Tr}\left[A^{\dagger}B\right]$

Autocorrelation function:

$$C(t) := \langle \mathcal{O}\mathcal{O}(t)\rangle = \left(\mathcal{O}\,|\,\mathcal{O}(t)\right) = \left(\mathcal{O}\,\middle|\,e^{it\mathcal{L}}\,\middle|\,\mathcal{O}\right) = \left(\mathcal{O}_0\,\middle|\,e^{it\mathcal{L}}\,\middle|\,\mathcal{O}_0\right) = \sum_{n=0}^{+\infty}\frac{(it)^n}{n!}\left(\mathcal{O}_0\,\middle|\,\mathcal{L}^n\,\middle|\,\mathcal{O}_0\right)$$

Defining moments μ_{2n} such that $\langle \mathcal{O}\mathcal{O}(t)\rangle = \sum_{n=0}^{+\infty}\frac{(it)^n}{n!}\mu_{2n}$:

$$\Longrightarrow \mu_{2n} = \left(\mathcal{O}_0\,\middle|\,\mathcal{L}^{2n}\,\middle|\,\mathcal{O}_0\right) = \left[\begin{pmatrix} 0 & b_1 & 0 & 0 & \dots & 0 \\ b_1 & 0 & b_2 & 0 & \dots & 0 \\ 0 & b_2 & 0 & b_3 & \dots & 0 \\ \vdots & & \ddots & & \ddots & \vdots \end{pmatrix}^{2n}\right]_{00}$$

$\longrightarrow$ Dyck paths.

$\longrightarrow$ Invertible $b_n \leftrightarrow \mu_{2n}$.

$\longrightarrow$ Analytical structure of $C(t)$.