Strong coupling of $\mathcal{N} = 2$ Superconformal field theories and holography

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Strings, Fields and Holograms

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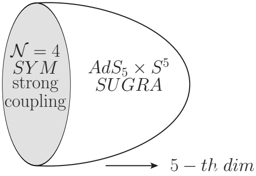
Based on [2109.00559] with M. Billò, M. Frau, A. Lerda, A. Pini

Motivations: AdS/CFT and exact results

- Maldacena realization represents a **milestone** in holographic context.
- Symmetries of $\mathcal{N} = 4$ SYM allow for **exact results** in the coupling $\lambda = g^2_{\mathrm{YM}} N$.



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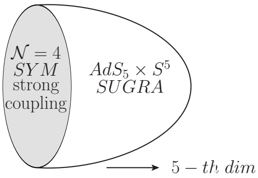
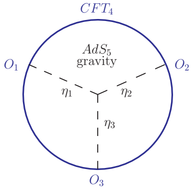


Example: three point functions of chiral operators $\langle O_1 O_2 O_3 \rangle$ [1].

- No quantum corrections on the gauge theory
- Three level $=$ strong coupling
- Match with the cubic term of SUGRA action on $AdS_5 \times S^5$



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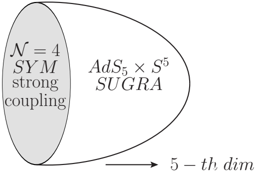

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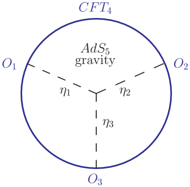

What about less symmetric $\mathcal{N} = 2$ superconformal case?

- In general harder to find exact results.
- We identify special theories where to obtain **exact results** for special observables and match them to a **dual gravity side**.

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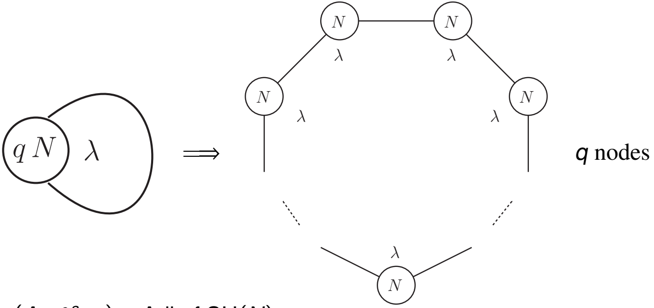
We consider a family of $\mathcal{N} = 2$ theories as $\mathbb{Z}_q$ orbifolds of $\mathcal{N} = 4$.



-  $= (A_\mu, \lambda^a, \varphi)_I$ Adj of $\mathrm{SU}(N)_I$
- —— $= (q, \bar{q}, \psi, \bar{\psi})$ $\left(\square, \bar{\square}\right)$ of $\mathrm{SU}(N)_I \times \mathrm{SU}(N)_{I+1}$
- Explicit Lagrangian formulation.
- They admit a holographic dual of type $AdS_5 \times S^5/\mathbb{Z}_q$.

Physical observables: chiral primary operators

Important class of operators in $\mathcal{N}=2$: **chiral primaries** $O_n(x) = \mathrm{tr}\,\varphi^n$

- $[\bar{Q}^A, O_n] = 0 \;\to\;$ 1/2 BPS.
- Protected by supersymmetry: $n = \frac{R}{2}$
- **Chiral ring** structure: OPE is non-singular $O_m(x)O_n(0) \sim C^{\ell}_{mn} O_\ell(0)$
- Choosing [2] $C^{\ell}_{mn} = \delta^{\ell}_{m+n}$, the 3-pt is written in terms of a 2-pt function:

$$\left\langle O_m(x_m) O_n(x_n) \bar{O}_\ell(x_\ell) \right\rangle = \frac{C_{mn\bar{\ell}}}{|x_{mn}|^{m+n-\ell} |x_{m\ell}|^{m+\ell-n} |x_{n\ell}|^{n+\ell-m}} \quad \left\langle O_m(x_m) \bar{O}_\ell(x_\ell) \right\rangle = \frac{g_{m\bar{\ell}}}{x_{m\ell}^{2m}}$$

$$C_{mn\bar{\ell}} = g_{m+n\,\bar{\ell}}$$

[2] [Baggio, Niarchos, Papadodimas, 2014-2016], [Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, Pufu, 2016]

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In quiver theories we have multiple vector multiplets: $O^{(I)}_n \sim \mathrm{tr}\,(\varphi^{(I)})^n$ and we concentrate on the 2-pt function:

$$\left\langle O^{(I)}_n(x)\;\bar{O}^{(J)}_n(0)\right\rangle \;=\; \frac{G^{(IJ)}_n(\lambda)}{x^{2n}}$$

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The path integral is mapped to a sphere S_4 and reduced to a finite dimensional integral [3]

$$Z_{\mathbb{R}^4} = \int \left[\mathcal{D}\Phi\right] \xrightarrow{\ Q\ } \mathcal{Z}_{S^4} = \int da_{\ [N\times N]}$$

$\mathbb{Z}_q$ orbifold partition function is reduced to a q-multimatrix model:

$$\mathcal{Z}_q = \int \prod_{I=1}^{q} da_I \, e^{-\frac{8\pi^2 N}{\lambda}\operatorname{tr} a_I^2 - \mathcal{S}_{\rm int}(a_I)} \,,$$

$$\mathcal{S}_{\rm int}(a_I) = \sum_{m=2}^{\infty}\sum_{k=0}^{2m}\frac{(-1)^{m+k}}{2m}\zeta(2m-1)\binom{2m}{k}\Big(\operatorname{tr} a_I^{2m-k} - \operatorname{tr} a_{I+1}^{2m-k}\Big)\Big(\operatorname{tr} a_I^{k} - \operatorname{tr} a_{I+1}^{k}\Big)$$

[3] [Pestun, 2007]

[4] [Billò, Fucito, Lerda, Morales, Stanev, Wen, FG, Gregori, 2017-2020], [F.G, Preti, 2020]

Mathematical tools: supersymmetric localization

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Expanding $S_{\rm int}$ in **perturbation theory**, $G_n^{(IJ)}(\lambda)$ computed in terms of Gaussian integrals [4].

$$G_n^{(IJ)}(\lambda) = \frac{1}{\mathcal{Z}_q}\prod_{I=1}^{q}\left\langle \left(1 - \mathcal{S}_{\rm int}(a_I) + \frac{1}{2!}\mathcal{S}_{\rm int}^2(a_I) + \dots\right) O_n^{(I)}(a_I) O_n^{(J)}(a_J) \right\rangle_{\rm Gaussian}$$

$$\text{Example}: G_2^{1,1}\big|_{q=4} = 1 - \frac{3\zeta(3)\lambda^2}{64\pi^4} + \frac{5\zeta(5)\lambda^3}{128\pi^6} + \frac{\left(27\zeta(3)^2 - 245\zeta(7)\right)\lambda^4}{8192\pi^8} + \frac{9(21\zeta(9) - 5\zeta(3)\zeta(5))\lambda^5}{8192\pi^{10}} + \dots$$

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Go beyond perturbation theory: untwisted and twisted operators

- We define untwisted and twisted operators as:

$$U_n = \frac{1}{\sqrt{q}}\Big(O_n^{(1)} + O_n^{(2)} + \ldots + O_n^{(q)}\Big)\,,$$

$$T_{\alpha,n} = \frac{1}{\sqrt{q}}\sum_{l=1}^{q}\rho^{-\alpha l}\,O_n^{(l)}\,,\quad \text{where } \alpha = 1,\ldots,q-1 \text{ and } \rho = \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}}$$

- Example for $\mathbb{Z}_2$: $U_n = \frac{1}{\sqrt{2}}\Big(O_n^{(1)} + O_n^{(2)}\Big)\,, \qquad T_n = \frac{1}{\sqrt{2}}\Big(O_n^{(1)} - O_n^{(2)}\Big)\,.$

- Special properties:
  - Orthogonality: $\big\langle U_n \overline{T}_{\alpha,n}\big\rangle = 0\,, \qquad \big\langle T_{\alpha,n}\overline{T}_{\beta,n}\big\rangle = 0 \text{ if } \alpha \neq \beta\,.$
  - Coupled to (un)twisted closed string states on $AdS_5 \times S^5/\mathbb{Z}_q$.

Effective Gaussian formulation in the planar limit

- We rewrite $\mathcal{Z}_q$ after the change of variables:

$$S_{\text{int}} = \sum_{\alpha=1}^{q-1} \left[4 s_\alpha \sum_{m=2}^{\infty} \sum_{k=2}^{2m} (-1)^{m+k} \binom{2m}{k} \frac{\zeta(2m-1)}{2m} \, T^\dagger_{\alpha,2m-k} \, T_{\alpha,k} \right], \qquad s_\alpha = \sin^2\left(\frac{\pi\alpha}{q}\right).$$

- Immediately we find: $\left\langle U_n \bar{U}_n \right\rangle = \left\langle O_n \bar{O}_n \right\rangle_{\mathcal{N}=4}$, same planar result as $\mathcal{N} = 4$.

- What about twisted sector?

  The interacting action is now **quadratic** for $T_{\alpha,n}$ variables! We rewrite the matrix model as an **effective Gaussian integral**:

$$\boldsymbol{z}_\alpha = \begin{pmatrix} T_{\alpha,2} \\ T_{\alpha,3} \\ \vdots \end{pmatrix}, \qquad \mathcal{Z}_\alpha = \int d^2 \boldsymbol{z}_\alpha \, \mathrm{e}^{-\boldsymbol{z}_\alpha^\dagger \boldsymbol{z}_\alpha + s_\alpha \boldsymbol{z}_\alpha^\dagger \mathsf{X} \boldsymbol{z}_\alpha} = \left[\det\left(\mathbb{1} - s_\alpha \mathsf{X} \right) \right]^{-1}.$$

- The full dependence on the coupling constant λ resides in an infinite dimensional matrix:

$$\mathsf{X}_{k,\ell} = -8(-1)^{k+\ell}\sqrt{k\,\ell}\int_0^\infty \frac{dt}{t}\,\frac{\mathrm{e}^t}{(\mathrm{e}^t-1)^2}\,J_k\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)J_\ell\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)\,.$$

- Twisted correlators can be written in a closed form, **exact in λ**:

$$\big\langle T_{\alpha,n}\,\overline{T}_{\alpha,n}\big\rangle = \frac{\det\Big[(\mathbb{1}-s_\alpha\,\mathsf{X}\,)^{-1}_{(n)}\Big]}{\det\Big[(\mathbb{1}-s_\alpha\,\mathsf{X}\,)^{-1}_{(n-1)}\Big]}\,,$$

- This formula can be expanded at **strong coupling** $\lambda\to\infty$ using Mellin transform methods:

$$\big\langle T_{\alpha,n}\,\overline{T}_{\alpha,n}\big\rangle \underset{\lambda\to\infty}{\sim} \frac{4\pi^2}{\lambda}\,\frac{1}{\sin^2\big(\frac{\pi\alpha}{q}\big)}\,n(n-1)\,.$$

Conclusion: holographic perspectives for $\mathcal{N}=2$ theories

STEP 1: match QFT strong coupling results with supergravity on $AdS_5 \times S^5/\mathbb{Z}_q$.

- The **untwisted states** have the same behavior as $\mathcal{N}=4$.
- Due to orbifold singularity, the **dual twisted states** $\eta_{\alpha,n}$ live on $AdS_5 \times S^1$ [5].
- Compactifying over S^1, the quadratic part of SUGRA action reads:

$$S[\eta] = \frac{1}{2}\sum_{\alpha=1}^{q-1}\sum_{n=2}^{\infty}\int_{\mathrm{AdS}_5} d^5x\,\sqrt{G_{\mathrm{AdS}_5}}\left[\frac{4\pi^2}{\lambda}\,\frac{1}{\sin^2\left(\frac{\pi\alpha}{q}\right)}\left(\partial\eta^*_{\alpha,n}\cdot\partial\eta_{\alpha,n} + n(n-4)\,\eta^*_{\alpha,n}\,\eta_{\alpha,n}\right)\right],$$

- The normalization perfectly matches our strong coupling results for 2-pt functions.

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FUTURE STEPS for AdS/CFT in $\mathcal{N} = 2$ settings:

- Complete the **3-pt functions** match.
- Go **beyond** the **protected spectrum**
- Explore **different** $\mathcal{N} = 2$ **theories** using similar techniques

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THANK YOU!

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